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Computation of a Primary Component of an Ideal from Its Associated Prime by Effective Localization

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Abstract

This is an enhanced full paper version of [Ishihara-Yokoyama, 2018] and contains detailed proofs, additional examples and new algorithms. In [Ishihara-Yokoyama, 2018], we proposed effective methods for localization of a polynomial ideal, which are called "Local Primary Algorithm (LPA)". Here, we consider the special case "localization by a prime ideal" and we introduce criteria for prime divisors and effective methods for computation of a primary component. For an ideal I and a prime ideal P, LPA computes a P-primary component of I after checking whether P is a prime divisor of I. It mainly uses *Double Ideal Quotient* (DIQ) $(I : (I : P))$ and its variants which contain useful information about localization of I. To examine its practicality, we compare it to another localization algorithm without DIQ. Based on computational experiments, we give further discussions about the practicality.

Keywords: Gröbner Basis, Primary Decomposition, Localization, Double Ideal Quotient

1 Introduction

The operation of "localization by a prime ideal" is widely known as a basic tool in commutative algebra and algebraic geometry. Here, we focus on computing a primary component from only its prime divisor and propose a new effective localization. As key notions, it uses *double ideal quotient* (DIQ) (and its variants) and *maximal independent set* (MIS).

We recall briefly the essence of [5]. Localization of ideals (as the saturation or the contraction of localized ideals) can be computed through its primary decomposition (see Remark 4), where algorithms of primary decomposition have been much studied in papers [2, 3, 7, 12]. However, in practice, the use of primary decomposition is not an efficient way since it tends to be very time-consuming. Hence, we focused on special localization (localization by a prime ideal) and compute a primary component directly, without its full primary decomposition. Then, we invented a direct method named *Local Primary Algorithm* (LPA) which computes a primary component, without its full primary decomposition. In more details, we explain some key points of LPA as follows.

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- LPA is based on several generating tools and criteria for primary components with different procedures for two cases; isolated and embedded.
- LPA uses *double ideal quotient and its variants* as tools for generating and checking primary components.
- *Double ideal quotient* (DIQ) is $(I : (I : J))$ for ideals I and J, which already appears in [14] to check associated primes or compute equidimensional hulls, and in [2], to compute equidimensional radicals.
- There are other important properties of DIQ and its variants toward effective localization. For instance, for ideals I, J and a primary decomposition $\mathcal{Q}$ of I, a variant of DIQ $(I : (I : J)^\infty)$ coincides with $\bigcap_{Q\in\mathcal{Q}, J\subset IK[X]_{\sqrt{Q}}\cap K[X]} Q$.

For practical implements we devised several efficient techniques for improving our LPA as follows (see [6, 14] for efficient computation of ideal quotient and saturation).

- ($P_G^{[m]}$-products) Use $P_G^{[m]} = (f_1^m, \ldots, f_r^m)$ for some generator $G = \{f_1, \ldots, f_r\}$ of P and the *equidimensional hull* (see Definition 10) hull$(I + P_G^{[m]})$ to compute a P-primary component, instead of using hull$(I + P^m)$ (see Lemma 54).
- (MIS-hull) Use *a maximal independent set* of P for computing hull$(\overline{Q})$ where $\overline{Q}$ is a P-hull-primary ideal (see Definition 13). Since a maximal independent set U of P is also a maximal independent set of $I + P^m$, we obtain hull$(I + P^m) = (I + P^m)K[X]_{K[U]^\times} \cap K[X]$ (see Lemma 58).
- (MIS-localization) Use a maximal independent set U of P at the first step of LPA to replace I for $IK[X]_{K[U]^\times} \cap K[X]$ (see Theorem 39).

As an enhanced full paper version of [5], this paper contains detailed proofs, additional examples and new algorithms. In particular, as additional development, we invent another localization algorithm using a well-known splitting tool of ideal instead of DIQ to compare it and the original LPAs (see Sect. 6). Furthermore, we make a new implementation on the computer algebra system Risa/Asir [11] and re-examine the performance in a number of examples in Sect. 7. As a reference, we show the timings of a full primary decomposition function `noro_pd.syci_dec` in Risa/Asir. Thanks to efficient techniques above, our experiment shows clearly the practicality of our direct localization method. From our experiments, we conclude that MIS-localization is the most efficient tool among our LPAs. However, there are some cases for which it is not efficient. Our main observation is the following;

- LPAs have strong effectiveness by its speciality.
- MIS-localization is much effective for many examples (see Table 1 and Table 2 in Sect. 7). However, its computational behavior is *unstable* (see Figures 2, 3 in Sect. 7).
- Effectiveness of the algorithms depends on ideals. At present, it is not predicable and thus it would be better to apply them in parallel.

This paper is organized as follows. Through Sect. 2 to Sect. 7, we add complete proofs and a lot of examples as an enhanced full paper version of [5]. In Sect. 2, we provide a mathematical basis for our criteria and algorithms. In Sect. 3, we introduce notions and properties of DIQ and its variants. In Sect. 4, we describe criteria for prime divisors and primary components by using DIQ and its variants. In Sect. 5, we explain LPA to compute the particular primary component without primary decomposition, after isolated and embedded prime divisor checks. In Sect. 6, the additional section, we generalize propositions in [5] and devise a new algorithm using splitting tool and maximal independent set instead of DIQ. In Sect. 7, we tested for many examples as experiments and discuss the behavior of each algorithm. In Sect. 8, we give some concluding remarks and the future works.

2 Mathematical Basis

Throughout this paper, we let K be a computable field (e.g. the rational field $\mathbb{Q}$ or a finite field), $X = \{x_1, \ldots, x_n\}$ a set of variables and $K[X] = K[x_1, \ldots, x_n]$ the polynomial ring. We write $(f_1, \ldots, f_t)_{K[X]}$ for the ideal generated by elements $f_1, \ldots, f_t$ in $K[X]$ and we simply use $(f_1, \ldots, f_t)$ if the ring is obvious. When we simply say I is an ideal, it means the I is an ideal of $K[X]$. Moreover, we denote the radical of I by $\sqrt{I}$.

2.1 Definition of Primary Decomposition and Localization

Here we give the definition of primary decomposition, which can be found in several books [1, 3, 6, 14].

Definition 1 (Primary Decomposition)
For an ideal I of $K[X]$, a set Q of primary ideals is called a primary decomposition *of I if $I = \bigcap_{Q \in \mathcal{Q}} Q$. A primary decomposition $\mathcal{Q} = \{Q_1, \ldots, Q_r\}$ is irredundant if the $\sqrt{Q_i}$ are all distinct and $Q_i \not\supset \bigcap_{j \neq i} Q_j$. We assume primary decomposition is irredundant. For a primary decomposition of I, each primary ideal is called* a primary component *of I. The prime ideal associated with a primary component of I is called* a prime divisor *of I. Among all prime divisors of I, minimal prime ideals are called* isolated prime divisors *of I and others are called* embedded prime divisors *of I. A primary component of I is called* isolated *if its prime divisor is isolated and* embedded *if its prime divisor is embedded. We denote by $\mathrm{Ass}(I)$ and $\mathrm{Ass}_{iso}(I)$ the set of all prime divisors of I and the set of all isolated prime divisors respectively.*

It is well-known that an isolated primary component does not depend on primary decompositions, while an embedded primary component does. From the perspective of algorithm, it tends to be more difficult to compute embedded primary components than isolated primary components.

We also give fundamental notions and properties related to a localization that can extract the particular primary components.

Definition 2 (Localization)
Let I be an ideal of $K[X]$ and S a multiplicatively closed set in $K[X]$. We call $IK[X]_S$ the localized ideal by S *and $IK[X]_S \cap K[X]$* the contraction of the localized ideal *respectively. For simplicity, we call the latter the* localization of I *with respect to S (see Definition 2.2 in [12]). For a multiplicatively closed set $K[X] \setminus P$, where P is a prime ideal, we denote it simply by $IK[X]_P \cap K[X]$. We assume a multiplicatively closed set S always does not contain* 0.

Example 3
In $\mathbb{Q}[X] = \mathbb{Q}[x, y]$, let $P = (x)$ be a prime ideal. For $S = \mathbb{Q}[X] \setminus P$ and $I = (x^2, xy)$, the localization of I by S is $I\mathbb{Q}[X]_S \cap \mathbb{Q}[X] = (x)$. For $P = (x, y)$ and $J = (x) \cap (x + 1) \cap (x + 2, y^2)$, the localization of J by P is $J\mathbb{Q}[X]_P \cap \mathbb{Q}[X] = (x)$.

We remark a relationship between primary decomposition and localization.

Remark 4 (Localization from Primary Decomposition)
Given a primary decomposition $\mathcal{Q}$ *of an ideal* I*, the localization of* I *by* S *is expressed as* $\bigcap_{Q\in\mathcal{Q},Q\cap S=\emptyset} Q$*. Moreover, it is also equal to* $(I : (\bigcap_{P\in \mathrm{Ass}(I),P\cap S\neq\emptyset} P)^{\infty})$*. Here, we are thinking mainly about* computable *multiplicatively closed set s.t. finitely generated one or the complement of a prime ideal. In these cases, we can decide efficiently whether* Q *and* S *intersect or not, by using Gröbner basis. Thus if we know all primary components or all associated primes, then we can compute localizations of* I *for any* computable *multiplicatively closed sets* S*. However, this method is not a direct method since it computes unnecessary primary components or associated primes.*

Next we introduce the notion of pseudo-primary ideal, which is an extension of the definition of primary ideal.

Definition 5 ([12], Definition 2.3)
Let Q *be an ideal. We say* Q *is* pseudo-primary *if* $\sqrt{Q}$ *is a prime ideal. In this case, we also say that* Q *is* $\sqrt{Q}$-pseudo-primary.

Example 6
Since $\sqrt{(x^2, xy)} = (x)$ *is a prime ideal, it follows that* (x^2, xy) *is an* (x)*-pseudo-primary ideal. Every* P*-primary ideal is* P*-pseudo-primary.*

With the notion of pseudo-primary ideal, we can define some special localization *the minimal* P*-pseudo-primary component* with respect to its isolated prime divisor P. It is equal to the intersection of all primary components whose radicals contain P but do not contain other isolated prime divisors.

Definition 7
Let I *be an ideal and* P *an isolated prime divisor of* I*. For a set of prime divisors*

$$\mathcal{P} = \{P' \in \mathrm{Ass}(I) \mid P \textit{ is the unique isolated prime divisor contained in } P'\}$$

and a multiplicatively closed set $S = K[X] \setminus \bigcup_{P'\in\mathcal{P}} P'$*, we call* $\overline{Q} = IK[X]_S \cap K[X]$ the minimal P-pseudo-primary component *of* I*. This definition is consistent with one in [12]. We note that the minimal* P*-pseudo-primary component is determined uniquely and has the* P*-isolated primary component of* I *as component. Also, every* P*-pseudo-primary component of* I *defined in [12] contains the minimal one defined here.*

Example 8
For $I = (x) \cap (x + 1) \cap (x^2, y) \subset \mathbb{Q}[x, y]$, (x^2, xy) *is the minimal* (x)*-pseudo-primary component of* I *and* $(x + 1)$ *is the minimal* $(x + 1)$*-pseudo-primary component of* I*.*

Remark 9
Every minimal P*-pseudo-primary component of* I *is a* P*-pseudo-primary ideal. Let* $\overline{Q}_P$ *be the minimal* P*-pseudo-primary component of* I*. Then* $I = \bigcap_{P\in\mathrm{Ass}_{iso}(I)} \overline{Q}_P \cap I'$ *for some* I' *s.t.* $\mathrm{Ass}_{iso}(I')\cap \mathrm{Ass}_{iso}(I) = \emptyset$*. This decomposition is called* a pseudo-primary decomposition *in [12], where it is computed by separators from given* $\mathrm{Ass}_{iso}(I)$*. Meanwhile, we introduce another method to compute* P*-pseudo-primary components by using double ideal quotient in Lemma 43.*

We may regard the minimal P-pseudo-primary component as a "column localization" since it has different dimensional primary components in general. Conversely, we may consider a "row localization", that contains equidimensional primary components.

Definition 10 ([2], Sect. 1)
Let I be an ideal and $\mathcal{Q}$ a primary decomposition of I. We call $\mathrm{hull}(I) = \bigcap_{Q\in\mathcal{Q},\dim(Q)=\dim(I)} Q$ the equidimensional hull *of I. Since every primary component Q satisfying* $\dim(Q) = \dim(I)$ *is isolated,* $\mathrm{hull}(I)$ *is determined independently from choice of primary decompositions.*

Example 11
For $I = (x) \cap (x + 1) \cap (x^2, y) \cap (x - 1, y) \subset \mathbb{Q}[x, y]$, *it follows that* $\mathrm{hull}(I) = (x) \cap (x + 1)$.

For a given I, $\mathrm{hull}(I)$ can be computed in several manners. For instance, it can be computed by Ext functors [2] or a regular sequence contained in I [14] as follows.

Proposition 12 ([2], Theorem 1.1. [14], Proposition 3.41)
Let I be an ideal and $u \subset I$ be a c-length regular sequence, where c is the codimension of I. Then $\mathrm{hull}(I) = ((u) : ((u) : I)) = \mathrm{ann}_{K[X]}(\mathrm{Ext}^c_{K[X]}(K[X]/I, K[X]))$.

Next, we introduce the notion of hull-primary ideal, which is an extension of the definition of pseudo-primary ideal. We use hull-primary ideal in Sec. 5.2.1 to devise practical techniques for LPA.

Definition 13 ([5], Definition 13)
Let I be an ideal. We say that I is hull-primary *if* $\mathrm{hull}(I)$ *is a primary ideal. For a prime ideal P, we say a hull-primary ideal I is* P-hull-primary *if* $P = \mathrm{hull}(\sqrt{I})$.

Example 14
Let $I = (x^2) \cap (x^3, y) \cap (x + 1, y + 1) \subset \mathbb{Q}[x, y]$. *Since* $\mathrm{hull}(I) = (x^2)$ *is (x)-primary, I is (x)-hull primary.*

As a pseudo-primary ideal has the unique isolated component, we obtain the following remark.

Remark 15
Every pseudo-primary ideal is hull-primary.

Using the following lemma and a variant of *double ideal quotient*, we can compute the isolated P-primary component of I in Section 5.

Lemma 16 ([5], Lemma 15)
Let P be an isolated prime divisor of I and $\overline{Q}_P$ the minimal P-pseudo-primary component of I. Then, $\overline{Q}_P$ is a P-hull-primary and $\mathrm{hull}(\overline{Q}_P)$ *is the isolated P-primary component of I.*

Proof By Remarks 9 and 15, it follows that $\overline{Q}_P$ is P-hull-primary and $\mathrm{hull}(\overline{Q}_P)$ is the isolated P-primary component. By the definition of $\overline{Q}_P$ and Lemma 72, we obtain that $\mathrm{hull}(\overline{Q}_P)$ is the isolated P-primary component of I. ∎

Example 17
Let $I = (x) \cap (x^2, y) \cap (x^2, y + 1) \subset \mathbb{Q}[x, y]$. *For* $P = (x)$, $\overline{Q}_P = (x^2, xy)$ *is the minimal P-pseudo primary component of I and* $\mathrm{hull}(\overline{Q}_P) = (x)$ *is the P-isolated primary component of I.*

2.2 Fundamental Properties of Ideal Quotient

We introduce fundamental properties of ideal quotient. The first two can be seen in several papers and books ([1], Lemma 4.4. [6], Lemma 4.1.3. [14], a remark before Proposition 3.56). The last two are direct consequences of the first two. We put a proof of Lemma 18 into Appendix.

Lemma 18 ([5], Lemma 19)
Let I and J be ideals, Q a primary ideal and $\mathcal{Q}$ a primary decomposition of I. Then,

$$(Q : J) = \begin{cases} Q & (J \not\subset \sqrt{Q}), \\ K[X] & (J \subset Q), \\ \sqrt{Q}\text{-primary ideal properly containing } Q & (J \not\subset Q, J \subset \sqrt{Q}), \end{cases} \tag{1}$$

$$(Q : J^\infty) = \begin{cases} Q & (J \not\subset \sqrt{Q}), \\ K[X] & (J \subset \sqrt{Q}), \end{cases} \tag{2}$$

$$(I : J) = \bigcap_{Q\in\mathcal{Q}, J\not\subset\sqrt{Q}} Q \cap \bigcap_{Q\in\mathcal{Q}, J\not\subset Q, J\subset\sqrt{Q}} (Q : J), \tag{3}$$

$$(I : J^\infty) = (I : \sqrt{J}^\infty) = \bigcap_{Q\in\mathcal{Q}, J\not\subset\sqrt{Q}} Q. \tag{4}$$

3 Double Ideal Quotient

Double Ideal Quotient (DIQ) is an ideal of shape $(I : (I : J))$ where I and J are ideals. For an ideal I and its primary decomposition $\mathcal{Q}$, we divide $\mathcal{Q}$ into three parts:

$$\mathcal{Q}_1(J) = \{Q \in \mathcal{Q} \mid J \not\subset \sqrt{Q}\},$$
$$\mathcal{Q}_2(J) = \{Q \in \mathcal{Q} \mid J \subset Q\},$$
$$\mathcal{Q}_3(J) = \{Q \in \mathcal{Q} \mid J \not\subset Q, J \subset \sqrt{Q}\}.$$

For example, letting $I = (x^2) \cap (x^3, y^2) \cap (y)$, $J = (x^2)$ and $\mathcal{Q} = \{(x^2), (x^3, y^2), (y)\}$ a primary decomposition of I, it follows that $\mathcal{Q}_1(J) = \{(y)\}$, $\mathcal{Q}_2(J) = \{(x^2)\}$, and $\mathcal{Q}_3(J) = \{(x^3, y^2)\}$.

Then, our DIQ is expressed precisely by components of them. The following proposition can be proved directly from Lemma 18.

Proposition 19 ([5], Proposition 20)
Let I and J be ideals. Then,

$$\begin{aligned}(I : (I : J)) &= \bigcap_{Q\in\mathcal{Q}_2(J)} \left(Q : (\bigcap_{Q'\in\mathcal{Q}_1(J)} Q' \cap \bigcap_{Q'\in\mathcal{Q}_3(J)} (Q' : J))\right) \\ &\quad \cap \bigcap_{Q\in\mathcal{Q}_3(J)} \left(Q : (\bigcap_{Q'\in\mathcal{Q}_1(J)} Q' \cap \bigcap_{Q'\in\mathcal{Q}_3(J)} (Q' : J))\right),\end{aligned} \tag{5}$$

$$\sqrt{(I : (I : J))} = \bigcap_{P\in\mathrm{Ass}(I), J\subset P} P. \tag{6}$$

Proof First, we show (5). We divide I into three parts:

$$I = \bigcap_{Q\in\mathcal{Q}_1(J)} Q \cap \bigcap_{Q\in\mathcal{Q}_2(J)} Q \cap \bigcap_{Q\in\mathcal{Q}_3(J)} Q.$$

Then,

$$\begin{aligned}(I:(I:J)) &= \left([\bigcap_{Q\in\mathcal{Q}_1(J)} Q \cap \bigcap_{Q\in\mathcal{Q}_2(J)} Q \cap \bigcap_{Q\in\mathcal{Q}_3(J)} Q] : (I:J)\right)\\ &= (\bigcap_{Q\in\mathcal{Q}_1(J)} Q:(I:J)) \cap (\bigcap_{Q\in\mathcal{Q}_2(J)} Q:(I:J)) \cap (\bigcap_{Q\in\mathcal{Q}_3(J)} Q:(I:J)).\end{aligned}$$

Since

$$(I:J) = \bigcap_{Q'\in\mathcal{Q}_1(J)} Q' \cap \bigcap_{Q'\in\mathcal{Q}_3(J)} (Q':J),$$

we obtain

- $$\begin{aligned}(\bigcap_{Q\in\mathcal{Q}_1(J)} Q:(I:J)) &= (\bigcap_{Q\in\mathcal{Q}_1(J)} Q:(\bigcap_{Q'\in\mathcal{Q}_1(J)} Q' \cap \bigcap_{Q'\in\mathcal{Q}_3(J)} (Q':J)))\\ &= K[X]\end{aligned}$$
- $$\begin{aligned}(\bigcap_{Q\in\mathcal{Q}_2(J)} Q:(I:J)) &= (\bigcap_{Q\in\mathcal{Q}_2(J)} Q:(\bigcap_{Q'\in\mathcal{Q}_1(J)} Q' \cap \bigcap_{Q'\in\mathcal{Q}_3(J)} (Q':J)))\\ &= \bigcap_{Q\in\mathcal{Q}_2(J)} (Q:(\bigcap_{Q'\in\mathcal{Q}_1(J)} Q' \cap \bigcap_{Q'\in\mathcal{Q}_3(J)} (Q':J)))\end{aligned}$$
- $$\begin{aligned}(\bigcap_{Q\in\mathcal{Q}_3(J)} Q:(I:J)) &= (\bigcap_{Q\in\mathcal{Q}_3(J)} Q:(\bigcap_{Q'\in\mathcal{Q}_1(J)} Q' \cap \bigcap_{Q'\in\mathcal{Q}_3(J)} (Q':J)))\\ &= \bigcap_{Q\in\mathcal{Q}_3(J)} (Q:(\bigcap_{Q'\in\mathcal{Q}_1(J)} Q' \cap \bigcap_{Q'\in\mathcal{Q}_3(J)} (Q':J))).\end{aligned}$$

The second property (6) can be proved directly from the property (5). ∎

This proposition can be used to prove the following criterion for prime divisors.

Corollary 20 ([14], Corollary 3.4)
Let I be an ideal and P a prime ideal. Then, P belongs to $\mathrm{Ass}(I)$ if and only if $P \supset (I:(I:P))$.

Proof We note $P \supset (I:(I:P))$ if and only if $P \supset \sqrt{(I:(I:P))}$. By Proposition 19, $\sqrt{(I:(I:P))} = \bigcap_{P'\in\mathrm{Ass}(I), P\subset P'} P'$. If $P \in \mathrm{Ass}(I)$, then $\sqrt{(I:(I:P))} = \bigcap_{P'\in\mathrm{Ass}(I), P\subset P'} P' \subset P$. On the other hand, if $P \supset \sqrt{(I:(I:P))}$, then there is $P' \in \mathrm{Ass}(I)$ s.t. $P' \subset P$ and $P' \supset P$. Thus $P = P' \in \mathrm{Ass}(I)$. ∎

Example 21
Let $I = (x^2, xy)$ in $\mathbb{Q}[x,y]$. Then, $P = (x)$ is a prime divisor of I and $(I:(I:P)) = (I:(x,y)) = (x) \subset P$.

Replacing ideal quotient with saturation in DIQ, we have the following variants.

Definition 22 (Variants of DIQ)
We call $(I:(I:J)^\infty)$ the first saturated quotient, $(I:(I:J^\infty)^\infty)$ the second saturated quotient, *and* $(I:(I:J^\infty))$ the third saturated quotient *respectively.*

In the following proposition, we can see that variants of DIQ have useful information about localization.

Proposition 23 ([5], Proposition 22)
Let $\mathcal{Q}$ be a primary decomposition of I. Then,

$$(I:(I:J)^{\infty}) = \bigcap_{Q\in\mathcal{Q}, J\subset IK[X]_{\sqrt{Q}}\cap K[X]} Q, \tag{7}$$

$$(I:(I:J^{\infty})^{\infty}) = \bigcap_{Q\in\mathcal{Q}, J\subset \sqrt{IK[X]_{\sqrt{Q}}\cap K[X]}} Q, \tag{8}$$

$$(I:(I:J^{\infty})) = \bigcap_{Q\in\mathcal{Q}_2(J)} (Q : \bigcap_{Q'\in\mathcal{Q}_1(J)} Q') \cap \bigcap_{Q\in\mathcal{Q}_3(J)} (Q : \bigcap_{Q'\in\mathcal{Q}_1(J)} Q'). \tag{9}$$

Proof Here, we give an outline of the proof. The formula (7) can be proved by combining the equation

$$(I:(I:J)^{\infty}) = (I:\sqrt{(I:J)}^{\infty}) = \bigcap_{Q\in\mathcal{Q}, \bigcap_{Q'\in\mathcal{Q}_1(J)} \sqrt{Q'}\cap\bigcap_{Q'\in\mathcal{Q}_3(J)} \sqrt{Q'}\not\subset\sqrt{Q}} Q$$

by Lemma 18 and the following equivalence

(1-a) $J\subset IK[X]_{\sqrt{Q}}\cap K[X]$.

(1-b) $\bigcap_{Q'\in\mathcal{Q}_1(J)} \sqrt{Q'}\cap\bigcap_{Q'\in\mathcal{Q}_3(J)} \sqrt{Q'}\not\subset \sqrt{Q}$.

for each $Q\in\mathcal{Q}$. The second formula (8) can be proved by combining the equation $(I:(I:J^{\infty})^{\infty}) = (I:(I:J^m)^{\infty}) = \bigcap_{Q\in\mathcal{Q},J^m\subset IK[X]_{\sqrt{Q}}\cap K[X]} Q$ for a sufficiently large m from the first formula (7), and the following equivalence

(2-a) $J^m\subset IK[X]_{\sqrt{Q}}\cap K[X]$ for a sufficiently large m.

(2-b) $J\subset \sqrt{IK[X]_{\sqrt{Q}}\cap K[X]}$.

for each $Q\in\mathcal{Q}$. The third formula (9) can be proved directly from Lemma 18.

Now, we explain some details. We show (1-a) implies (1-b). If

$$\bigcap_{Q'\in\mathcal{Q}_1(J)} \sqrt{Q'}\cap\bigcap_{Q'\in\mathcal{Q}_3(J)} \sqrt{Q'}\subset\sqrt{Q},$$

then by Lemma 85, $\sqrt{Q'}\subset\sqrt{Q}$ for some $Q'\in\mathcal{Q}_1(J)\cup\mathcal{Q}_3(J)$. Since $Q'\subset\sqrt{Q'}\subset\sqrt{Q}$, we obtain $IK[X]_{\sqrt{Q}}\cap K[X] = \bigcap_{Q''\in\mathcal{Q},Q''\subset\sqrt{Q}} Q''\subset Q'$. However, since $Q'\in\mathcal{Q}_1(J)\cup\mathcal{Q}_3(J)$, we obtain $J\not\subset Q'$ and this contradicts $J\subset IK[X]_{\sqrt{Q}}\cap K[X]\subset Q'$.

Show (1-b) implies (1-a). Let $Q'\in\mathcal{Q}$ contained $\sqrt{Q}$. Since $\bigcap_{Q''\in\mathcal{Q}_1(J)} \sqrt{Q''}\cap\bigcap_{Q''\in\mathcal{Q}_3(J)} \sqrt{Q''}\not\subset \sqrt{Q}$, we obtain $Q'\notin\mathcal{Q}_1(J)\cup\mathcal{Q}_3(J)$ and $Q'\in\mathcal{Q}_2(J)$. Hence, $J\subset Q'$ and $J\subset\bigcap_{Q'\subset\sqrt{Q}} Q' = IK[X]_{\sqrt{Q}}\cap K[X]$.

Trivially, (2-a) implies (2-b) since $J\subset\sqrt{J^m}\subset\sqrt{IK[X]_{\sqrt{Q}}\cap K[X]}$. Show (2-b) implies (2-a). For $Q\in\mathcal{Q}_2(J)\cup\mathcal{Q}_3(J)$, let $m_Q = \min\{m \mid J^m\subset Q\}$ and $m = \max\{m_Q \mid Q\in\mathcal{Q}_2(J)\cup\mathcal{Q}_3(J)\}$. Then, $(I:J^{\infty}) = (I:J^m)$. Since $IK[X]_{\sqrt{Q}}\cap K[X] = \bigcap_{Q'\in\mathcal{Q},Q'\subset\sqrt{Q}} Q'$, we obtain $Q'\in\mathcal{Q}_2(J)\cup\mathcal{Q}_3(J)$ for any $Q'\in\mathcal{Q}$ contained in $\sqrt{Q}$. Thus, we obtain $J^m\subset IK[X]_{\sqrt{Q}}\cap K[X]$.

Finally, we show (9). Since $(I : J^\infty) = \bigcap_{Q' \in \mathcal{Q}_1(J)} Q'$, we obtain

$$\begin{aligned}(I : (I : J^\infty)) &= (I : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q') \\ &= (\bigcap_{Q \in \mathcal{Q}_1(J)} Q \cap \bigcap_{Q \in \mathcal{Q}_2(J)} Q \cap \bigcap_{Q \in \mathcal{Q}_3(J)} Q : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q') \\ &= \bigcap_{Q \in \mathcal{Q}_2(J)} (Q : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q') \cap \bigcap_{Q \in \mathcal{Q}_3(J)} (Q : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q').\end{aligned}$$

∎

Example 24
For $I = (x^2) \cap (x^3, y^2) \cap (x^4, y^3, z^2) \cap (z)$ and $J = (x^2)$,

$$\begin{aligned}(I : (I : J)^\infty) &= \bigcap_{Q \in \mathcal{Q}, J \subset IK[X]_{\sqrt{Q}} \cap K[X]} Q = (x^2), \\ (I : (I : J^\infty)^\infty) &= \bigcap_{Q \in \mathcal{Q}, J \subset \sqrt{IK[X]_{\sqrt{Q}} \cap K[X]}} Q = (x^2) \cap (x^3, y^2), \\ (I : (I : J^\infty)) &= \bigcap_{Q \in \mathcal{Q}_2(J)} (Q : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q') \cap \bigcap_{Q \in \mathcal{Q}_3(J)} (Q : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q') = (x^2) \cap (x^3, y^2) \cap (x^4, y^3, z).\end{aligned}$$

Using the first saturated quotient, we devise criteria for primary components in Section 4. The second saturated quotient can be used to an isolated prime divisors check and generate an isolated primary component in Section 5. The third saturated quotient gives another prime divisor criterion (Criterion 5 in Section 4) by the following proposition.

Proposition 25 ([5], Proposition 23)
Let I and J be ideals. Then

$$\sqrt{(I : (I : J^\infty))} = \bigcap_{P \in \mathrm{Ass}(I), J \subset P} P.$$

In particular, $\sqrt{(I : (I : J))} = \sqrt{(I : (I : J^\infty))}$.

Proof Let $\mathcal{Q}$ be a primary decomposition of I. By Proposition 23 (9),

$$\sqrt{(I : (I : J^\infty))} = \bigcap_{Q \in \mathcal{Q}_2(J)} \sqrt{(Q : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q')} \cap \bigcap_{Q \in \mathcal{Q}_3(J)} \sqrt{(Q : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q')}.$$

Since $\mathcal{Q}$ is minimal, we obtain $Q \not\supset \bigcap_{Q' \in \mathcal{Q}_1(J)} Q'$ for any $Q \in \mathcal{Q}_2(J)$ and $Q \not\supset \bigcap_{Q' \in \mathcal{Q}_1(J)} Q'$ for any $Q \in \mathcal{Q}_3(J)$. Thus, by Lemma 18,

$$\begin{aligned}\sqrt{(I : (I : J^\infty))} &= \textstyle\bigcap_{Q \in \mathcal{Q}_2(J)} \sqrt{(Q : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q')} \cap \bigcap_{Q \in \mathcal{Q}_3(J)} \sqrt{(Q : \bigcap_{Q' \in \mathcal{Q}_1(J)} Q')} \\ &= \textstyle\bigcap_{Q \in \mathcal{Q}_2(J)} \sqrt{Q} \cap \bigcap_{Q \in \mathcal{Q}_3(J)} \sqrt{Q} = \bigcap_{P \in \mathrm{Ass}(I), J \subset P} P.\end{aligned}$$

From (6) in Proposition 19, we obtain $\sqrt{(I : (I : J))} = \sqrt{(I : (I : J^\infty))}$. ∎

Example 26
For $I = (x^2) \cap (x^3, y^2) \cap (y)$ and $J = (x^2)$, $\sqrt{(I : (I : J^\infty))} = \bigcap_{P \in \mathrm{Ass}(I), J \subset P} P = (x)$.

4 Criteria for Primary Component and Prime Divisor

In this section, we present several criteria for primary component which check whether a P-primary ideal Q is a primary component of I or not without computing primary decomposition of I, based on the first saturated quotient. We first propose a general criterion applicable to any primary ideals. Later, we propose some specialized criteria aiming for isolated primary components and maximal ones. Finally, we add criteria for prime divisors.

4.1 General Primary Component Criterion

We use the first saturated quotient to check whether a given primary ideal is a component or not. We introduce a key notion *saturated quotient invariant*.

Definition 27 ([5], Definition 24)
Let I and J be ideals. We say that J is saturated quotient invariant *of I if $(I : (I : J)^\infty) = J$.*

Example 28
Let $I = (x) \cap (x^2, y)$ and $J = (x)$. Then J is saturated quotient invariant of I since $(I : (I : J)^\infty) = (I : (x, y)^\infty) = (x)$.

Any localization of ideal is saturated quotient invariant of the ideal. Conversely, any proper saturated quotient invariant ideal of I is some localization of I.

Lemma 29 ([5], Lemma 25)
Let I be an ideal and J a proper ideal of $K[X]$. Then, the following conditions are equivalent.
(A) $J = IK[X]_S \cap K[X]$ for some multiplicatively closed set S.
(B) J is saturated quotient invariant of I.

Proof Let $\mathcal{Q}$ be a primary decomposition. Show (A) implies (B). From Proposition 23 (7),

$$(I : (I : IK[X]_S \cap K[X])^\infty) = \bigcap_{Q\in\mathcal{Q}, IK[X]_S\cap K[X]\subset IK[X]_{\sqrt{Q}}\cap K[X]} Q. \tag{10}$$

By Lemma 77, $IK[X]_S \cap K[X] \subset IK[X]_{\sqrt{Q}} \cap K[X]$ if and only if $Q \cap S = \emptyset$. Thus,

$$\bigcap_{Q\in\mathcal{Q}, IK[X]_S\cap K[X]\subset IK[X]_{\sqrt{Q}}\cap K[X]} Q = \bigcap_{Q\in\mathcal{Q}, Q\cap S=\emptyset} Q, \tag{11}$$

Combining (10), (11) and $IK[X]_S \cap K[X] = \bigcap_{Q\in\mathcal{Q},Q\cap S=\emptyset} Q$ by Remark 4, we obtain $(I : (I : IK[X]_S \cap K[X])^\infty) = IK[X]_S \cap K[X]$.

Next, show (B) implies (A). From Proposition 23 (7),

$$(I : (I : J)^\infty) = \bigcap_{J\subset IK[X]_{\sqrt{Q}}\cap K[X]} Q = J. \tag{12}$$

Let $\mathcal{P} = \{\sqrt{Q} \mid Q \in \mathcal{Q}, J \subset IK[X]_{\sqrt{Q}} \cap K[X]\}$. We may assume $\mathcal{P} \neq \emptyset$, otherwise $\mathcal{P} = \emptyset$ and $J = K[X]$. Then $\mathcal{P}$ is *an isolated set* (see Definition 74) since if $P' \in \mathrm{Ass}(I)$ and $P' \subset P$ for some $P \in \mathcal{P}$, then $J \subset IK[X]_P \cap K[X] \subset IK[X]_{P'} \cap K[X]$ and $P' \in \mathcal{P}$. Let $S = K[X] \setminus \bigcup_{P\in\mathcal{P}} P$. By Lemma 75, $IK[X]_S \cap K[X] = \bigcap_{Q\in\mathcal{Q}, \sqrt{Q}\in\mathcal{P}} Q = \bigcap_{J\subset IK[X]_{\sqrt{Q}}\cap K[X]} Q$. By (12), we obtain $IK[X]_S \cap K[X] = J$.
∎

Example 30
Let $I = (x) \cap (x^2, y)$ and $J = (x)$. Then J is saturated quotient invariant of I and $J = IK[X]_{(x)} \cap K[X]$.

Based on Lemma 29, we have the following criterion for primary component.

Theorem 31 (Criterion 1. [5], Theorem 26)
Let I be an ideal and P a prime divisor of I. For a P-primary ideal Q, if $Q \not\supset (I : P^\infty)$, then the following conditions are equivalent.
(A) Q is a P-primary component for some primary decomposition of I.
(B) $(I : P^\infty) \cap Q$ is saturated quotient invariant of I.

Proof Show (A) implies (B). Let $\mathcal{Q}$ be a primary decomposition. Let $\mathcal{P} = \{P' \in \mathrm{Ass}(I) \mid P \not\subset P' \text{ or } P' = P\}$ and $S = K[X] \setminus \bigcup_{P' \in \mathcal{P}} P'$. Then S is a multiplicatively closed set and $(I : P^\infty) \cap Q \subset IK[X]_S \cap K[X]$ since $(I : P^\infty) \cap Q = \bigcap_{Q' \in \mathcal{Q}, P \not\subset \sqrt{Q'}} Q' \cap Q$. For each $Q' \in \mathcal{Q}$ with $Q' \cap S = \emptyset$, there is $P' \in \mathcal{P}$ such that $\sqrt{Q'} \subset P'$, i.e. $\sqrt{Q'} \in \mathcal{P}$. Thus, $(I : P^\infty) \cap Q \supset IK[X]_S \cap K[X]$ and $(I : P^\infty) \cap Q = IK[X]_S \cap K[X]$. By Lemma 29, $IK[X]_S \cap K[X]$ is saturated quotient invariant of I.

Show (B) implies (A). By Lemma 29, there is a multiplicatively closed set S such that $(I : P^\infty) \cap Q = IK[X]_S \cap K[X]$. Let $\mathcal{Q}$ be a primary decomposition of I. We know $IK[X]_S \cap K[X] = \bigcap_{Q' \in \mathcal{Q}, Q' \cap S = \emptyset} Q'$. By the assumption, $Q \not\supset (I : P^\infty)$ and thus $(I : P^\infty) \cap Q$ has a P-primary component. Then neither $\bigcap_{Q' \in \mathcal{Q}, Q' \cap S \neq \emptyset} Q'$ nor $(I : P^\infty)$ has a P-primary component. Hence,

$$I = (I : P^\infty) \cap Q \cap \bigcap_{Q' \in \mathcal{Q}, Q' \cap S \neq \emptyset} Q' = \bigcap_{Q' \in \mathcal{Q}, P \not\subset \sqrt{Q'}} Q' \cap Q \cap \bigcap_{Q' \in \mathcal{Q}, Q' \cap S \neq \emptyset} Q'$$

is a primary decomposition and Q is its P-primary component. ∎

Example 32
Let $I = (x) \cap (x^2, y^2) \cap (x^3, y^3, z) \cap (y) \cap (x + 1, z)$ and $P = (x, y)$ in $\mathbb{Q}[x, y, z]$. Then, $(I : P^\infty) = (x) \cap (y) \cap (x + 1, z)$. We think the following two P-primary ideals.

- *$Q_1 = (x^2, y^2)$. Since $Q_1 \not\supset (I : P^\infty)$ and $(I : (I : ((I : P^\infty) \cap Q_1))^\infty) = (x) \cap (y) \cap (x + 1, z) \cap (x^2, y^2) = (I : P^\infty) \cap Q_1$, we obtain (x^2, y^2) is a P-primary component of I.*
- *$Q_2 = (x^2, x + y)$. Since $(I : (I : ((I : P^\infty) \cap Q_2))^\infty) = (x) \cap (y) \cap (x + 1, z) \neq (I : P^\infty) \cap Q_2$, we obtain $(x^2, x + y)$ is not a P-primary component of I.*

4.2 Other Criteria for Primary Component

Next, we propose criteria for primary components having special properties which can be applied for particular prime divisors. These criteria may be computed more easily than the general one.

4.2.1 Criterion for Isolated Primary Component:

If Q is a primary ideal whose radical is an isolated divisor P of an ideal I, then we don't need to compute $(I : P^\infty)$ in Theorem 31 since the P-primary component of I is the localization of I by P.

Theorem 33 (Criterion 2. [5], Theorem 27)
Let I be an ideal and P an isolated prime divisor of I. For a P-primary ideal Q, the following conditions are equivalent.
(A) Q is the isolated P-primary component of I.
(B) $(I : (I : Q)^\infty) = Q$.

Proof Show (A) implies (B). Let $S = K[X] \setminus P$. By Lemma 29, $Q = IK[X]_S \cap K[X]$ is saturated quotient invariant of I and thus $(I : (I : Q)^\infty) = Q$. Next, we show (B) implies (A). By Lemma 29, there is a multiplicatively closed set S s.t. $IK[X]_S \cap K[X] = Q$. Since Q is primary, $IK[X]_S \cap K[X]$ is the isolated P-primary component. ∎

Example 34
For $I = (x^2) \cap (x^3, y^2) \cap (y)$, a primary component $Q = (x^2)$ is isolated and $(I : (I : Q)^\infty) = (x^2) = Q$.

4.2.2 Criterion for Maximal Primary Component:

Each isolated prime divisor is minimal in $\mathrm{Ass}(I)$. On the contrary, we consider "maximal prime divisor" and propose the following criterion for it.

Definition 35
Let P be a prime divisor of I. We say P is maximal *if there is no prime divisor P' of I containing P properly.*

Example 36
For $I = (x) \cap (x^2, y^2) \cap (z^2)$ in $\mathbb{Q}[x, y, z]$, prime divisors $P_1 = (x, y)$ and $P_2 = (z)$ are maximal in $\mathrm{Ass}(I) = \{(x), (x, y), (z)\}$.

Theorem 37 (Criterion 3. [5], Theorem 29)
Let I be an ideal and P a maximal prime divisor of I. For P-primary ideal Q, the following conditions are equivalent.
(A) *Q is a P-primary component of I.*
(B) $(I : P^\infty) \cap Q = I$.

Proof Show (A) implies (B). Let $\mathcal{Q}$ be a primary decomposition of I with $Q \in \mathcal{Q}$. Since P is maximal in $\mathrm{Ass}(I)$, $(I : P^\infty) = \bigcap_{Q' \in \mathcal{Q}, \sqrt{Q'} \not\supset P} Q' = \bigcap_{Q' \in \mathcal{Q}, Q' \neq Q} Q'$. Thus, $(I : P^\infty) \cap Q = \bigcap_{Q' \in \mathcal{Q}, Q' \neq Q} Q' \cap Q = I$. Next, we show (B) implies (A). Let $\mathcal{Q}'$ be a primary decomposition of $(I : P^\infty)$. Since $\mathcal{Q}'$ does not have P-primary component, $\mathcal{Q}' \cup \{Q\}$ is a primary decomposition of I. ∎

Example 38
Let $I = (x) \cap (x^2, y^2) \cap (z^2)$ and $P = (x, y)$ in $\mathbb{Q}[x, y, z]$. Then P is maximal in $\mathrm{Ass}(I)$ *and $Q = (x^2, y^2)$ is a P-primary component of I since $(I : P^\infty) \cap Q = (x) \cap (z^2) \cap (x^2, y^2) = I$.*

4.2.3 Criterion for Another General Primary Component:

The general case can be reduced to maximal case via localization by maximal independent set. A subset U of X is called a maximal independent set of I if $K[U] \cap I = 0$ and the cardinality of U is equal to the dimension of I (see [6] for its computation). Letting $S = K[U]^\times = K[U] \setminus \{0\}$, we obtain the following as a special case of Lemma 72.

Theorem 39 (Criterion 4. [5], Theorem 30)
Let I be an ideal and P a prime divisor of I. If U is a maximal independent set of P in X and Q is a P-primary ideal , then the following conditions are equivalent.
(A) *Q is a primary component of I.*
(B) *Q is a primary component of $IK[X]_{K[U]^\times} \cap K[X]$.*

Example 40
For $I = (x) \cap (x^2, y) \cap (x^3, y^2, z)$, we obtain (x^2, y) is a primary component of both I and $I\mathbb{Q}[X]_{(x,y)} \cap \mathbb{Q}[X] = (x) \cap (x^2, y)$.

4.3 Additional Criterion for Prime Divisor

Here, we add a criterion for prime divisor based on the third saturated quotient.

Theorem 41 (Criterion 5. [5], Theorem 31)
Let I be an ideal and P a prime ideal. Then, the following conditions are equivalent.
(A) $P \in \mathrm{Ass}(I)$.
(B) $P \supset (I : (I : P))$.
(C) $P \supset (I : (I : P^\infty))$.

Proof By Corollary 20, (A) is equivalent to (B). By Proposition 25, $\sqrt{(I : (I : P))} = \sqrt{(I : (I : P^\infty))} = \bigcap_{P' \in \mathrm{Ass}(I), P \subset P'} P'$. Thus, equivalence between (A) and (C) is proved by the similar way of Corollary 20. ∎

Example 42
For $I = (x^2) \cap (x^4, y) \cap (x+1)$ and a prime divisor $P = (x)$, we obtain $(I : (I : P)) = (x) \subset P$ and $(I : (I : P^\infty)) = (x^2) \cap (x^4, y) \subset P$.

Next, we devise another way to compute pseudo-primary components and criteria for isolated prime divisors based on the second saturated quotient.

Lemma 43 ([5], Lemma 32)
Let I be an ideal and P an isolated prime divisor of I. If $\overline{Q}$ is the minimal P-pseudo-primary component of I, then $(I : (I : P^\infty)^\infty) = \overline{Q}$.

Proof Let $\mathcal{Q}$ be a primary decomposition of I. By Proposition 23 (8),

$$(I : (I : P^\infty)^\infty) = \bigcap_{Q \in \mathcal{Q}, P \subset \sqrt{IK[X]_{\sqrt{Q}} \cap K[X]}} Q.$$

Thus it is enough to show that the following statements are equivalent for each $Q \in \mathcal{Q}$.
(1-a) $P \subset \sqrt{IK[X]_{\sqrt{Q}} \cap K[X]}$.
(1-b) P is the unique isolated prime divisor which is contained in $\sqrt{Q}$.
Show (1-a) implies (1-b). As $\sqrt{IK[X]_{\sqrt{Q}} \cap K[X]} \subset \sqrt{Q}$, we know $P \subset \sqrt{Q}$. Then, suppose there is another isolated prime divisor P' contained in $\sqrt{Q}$. We obtain

$$\sqrt{IK[X]_{\sqrt{Q}} \cap K[X]} = \bigcap_{Q' \in \mathcal{Q}, Q' \subset \sqrt{Q}} \sqrt{Q'} \subset P'.$$

However, this implies $P \subset P'$ and contradicts that P' is isolated. It is easy to prove that (1-b) implies (1-a). Since P is the unique isolated prime divisor which is contained in $\sqrt{Q}$, we obtain that

$$\sqrt{IK[X]_{\sqrt{Q}} \cap K[X]} = \bigcap_{Q' \in \mathcal{Q}, Q' \subset \sqrt{Q}} \sqrt{Q'} = P.$$

∎

Example 44
For $I = (x) \cap (x^2, y^2) \cap (y+1)$ and $P = (x)$, we obtain $(I : (I : P^\infty)^\infty) = (x) \cap (x^2, y^2)$ is the minimal P-pseudo-primary component of I.

Using Lemma 43, we obtain the following criterion for isolated prime divisor.

Theorem 45 (Criterion 6. [5], Theorem 33)
Let I be an ideal and P a prime ideal containing I. Then, the following conditions are equivalent.
(A) P is an isolated prime divisor of I.
(B) $(I : (I : P^\infty)^\infty) \neq K[X]$.

Proof Show (A) implies (B). By Lemma 43, $(I : (I : P^\infty)^\infty) = \overline{Q} \neq K[X]$. Show (B) implies (A). By Proposition 23 (8),

$$(I : (I : P^\infty)^\infty) = \bigcap_{Q \in \mathcal{Q}, P \subset \sqrt{IK[X]_{\sqrt{Q}} \cap K[X]}} Q \neq K[X]$$

for a primary decomposition $\mathcal{Q}$ of I. Then, there is an isolated prime divisor P' containing P. Since $\sqrt{I} \subset P \subset P'$ and P' is isolated, this implies $P = P'$ is isolated. ∎

Since each prime divisor of I contains I, Theorem 45 directly induces the following.

Corollary 46 (Criterion 7. [5], Corollary 34)
Let I be an ideal and P a prime divisor of I. Then,
(i) *P is isolated if $(I : (I : P^\infty)^\infty) \neq K[X]$,*
(ii) *P is embedded if $(I : (I : P^\infty)^\infty) = K[X]$.*

Example 47
Let $I = (x) \cap (x^2, y^2) \cap (y+1)$. For a prime divisor $P_1 = (x)$, $(I : (I : P^\infty)^\infty) = (x) \cap (x^2, y^2) \neq \mathbb{Q}[X]$ and P_1 is isolated. For a prime divisor $P_2 = (x, y)$, $(I : (I : P^\infty)^\infty) = \mathbb{Q}[X]$ and P_2 is embedded.

5 Local Primary Algorithm

In this section, we devise Local Primary Algorithm (LPA) which computes P-primary component of I. Our method applies different procedures for two cases; isolated and embedded. Algorithm 1 shows the outline of LPA. Its termination comes from Proposition 48. We remark that, for given prime divisors disjoint from a multiplicatively closed set S, we can compute all primary components disjoint from S by LPA. Then their intersection gives the localization by S.

5.1 Generating Primary Component

First, we introduce several ways to generate primary components through equidimensional hull computation.

Proposition 48 ([2], Section 4. [10], Remark 10)
Let I be an ideal and P a prime divisor of I. For any positive integer m, $I + P^m$ is P-hull-primary, and for a sufficiently large integer m, $\mathrm{hull}(I + P^m)$ is a P-primary component appearing in a primary decomposition of I.

Example 49
For $I = (x) \cap (x^2, y) \cap (x^3, y^2, z)$ and $P = (x, y)$, we obtain $I + P^3 = (x^3, x^2y, xy^2, y^3, x^2z, xyz)$ and $\mathrm{hull}(I + P^3) = (x^2, xy, y^3)$ is a P-primary component of I.

We can use Criteria for Primary Component to check m is large enough or not. If P is an isolated prime divisor, then the component is computed directly by using the second saturated quotient. By Lemma 16 and Lemma 43, we obtain the following theorem. To compute equidimensional hull, we can use regular sequence (see Proposition 12) or maximal independent set (see Lemma 58).

Theorem 50 ([5], Theorem 36)
Let I be an ideal and P an isolated prime divisor of I. Then

$$\mathrm{hull}((I : (I : P^\infty)^\infty))$$

is the isolated P-primary component of I.

Example 51
For $I = (x^2) \cap (x^3, y^2) \cap (y + 1)$ and $P = (x)$, the isolated P-primary component is $\mathrm{hull}((I : (I : P^\infty)^\infty)) = \mathrm{hull}((x^2) \cap (x^3, y^2)) = (x^2)$.

Algorithm 1 General Frame of Local Primary Algorithm

Input: I: an ideal, P: a prime ideal
Output: • a P-primary component of I if P is a prime divisor of I
• "P is not a prime divisor" otherwise

if P is a prime divisor of I (**Criterion 5**) **then**
  if P is isolated (**Criteria 6,7**) **then**
    $\overline{Q} \leftarrow$ the minimal P-pseudo-primary component of I (**Lemma 43**)
    $Q \leftarrow \mathrm{hull}(\overline{Q})$ (**Theorem 50**)
    return Q is the isolated P primary component
  else
    $m \leftarrow 1$, $Q \leftarrow K[X]$
    while Q is not primary component of I (**Criteria 1,3,4**) **do**
      $\overline{Q} \leftarrow$ a P-hull-primary ideal related to m (**Proposition 48, Lemma 54**)
      $Q \leftarrow \mathrm{hull}(\overline{Q})$
      $m \leftarrow m + 1$
    end while
    return Q is an embedded P-primary component
  end if
else
  return "P is not a prime divisor"
end if

5.2 Techniques for Improving LPA

We introduce practical techniques for implementing LPA.

5.2.1 Another Way of Generating Primary Component

Let $G = \{f_1, \dots, f_r\}$ be a generator of a prime ideal P. Usually we take $\{f_1^{e_1} f_2^{e_2} \cdots f_r^{e_r} \mid e_1 + \cdots + e_r = m\}$ as a generator of P^m for a positive integer m. However, this generator has $\frac{(r+m-1)!}{(r-1)!m!}$ elements and it becomes difficult to compute $\mathrm{hull}(I + P^m)$ when m becomes large. To avoid the explosion of the number of the generator, we can use $P_G^{[m]} = (f_1^m, \dots, f_r^m)$ instead.

First, we introduce a proposition to compute primary decomposition by using equdimensional hull.

Lemma 52 ([5], Lemma 37)
Let $\mathcal{Q}$ be a primary decomposition of I and $Q \in \mathcal{Q}$. If $\sqrt{Q}$-hull-primary ideal Q' satisfies $I \subset Q' \subset Q$, then $(\mathcal{Q} \setminus \{Q\}) \cup \{\mathrm{hull}(Q')\}$ is another primary decomposition of I.

Proof By Lemma 81, we obtain $I \subset Q' \subset \mathrm{hull}(Q') \subset Q$. Since $I \cap \mathrm{hull}(Q') = I$ and $Q \cap \mathrm{hull}(Q') = \mathrm{hull}(Q')$, we obtain

$$I = I \cap \mathrm{hull}(Q') = \left(\bigcap_{Q'' \in \mathcal{Q}, Q'' \neq Q} Q'' \cap Q \right) \cap \mathrm{hull}(Q') = \bigcap_{Q'' \in \mathcal{Q}, Q'' \neq Q} Q'' \cap \mathrm{hull}(Q').$$

Thus, $(\mathcal{Q} \setminus \{Q\}) \cup \{\mathrm{hull}(Q')\}$ is an irredundant primary decomposition of I. ∎

Example 53
Let $I = (x) \cap (x^2, y) \cap (z)$, $Q' = (x^2, xy, y^2) \cap (x^2, xy, y^3, z + 1)$ and $P = (x, y)$. Then, Q' is P-hull-primary. For a primary component $Q = (x^2, y)$, we obtain $I \subset Q' \subset Q$ and $\mathrm{hull}(Q') = (x^2, xy, y^2)$ is a P-primary component of I.

Next, the following lemma gives another efficient way to compute a primary component from its prime divisor.

Lemma 54 ([5], Lemma 38)
For any positive integer m, $I + P_G^{[m]}$ is P-hull-primary, and for a sufficiently large m, $\mathrm{hull}(I + P_G^{[m]})$ is a P-primary component appearing in a primary decomposition of I if P is a prime divisor of I.

Proof As $\sqrt{P_G^{[m]}} = P$ and $\sqrt{I + P} = \sqrt{I + P_G^{[m]}} = P$, $I + P_G^{[m]}$ is P-hull-primary. By Proposition 48, $\mathrm{hull}(I + P^m)$ is a P-primary component of I for a sufficiently large m. Since $I \subset I + P_G^{[m]} \subset I + P^m \subset \mathrm{hull}(I + P^m)$, $\mathrm{hull}(I + P_G^{[m]})$ is a P-primary component by Lemma 52. ∎

Example 55
For $I = (x) \cap (x^2, y) \cap (x^3, y^2, z)$ and $P = (G) = (x, y)$, we obtain $I + P_G^{[3]} = (x^3, xy^2, y^3, x^2z, xyz)$ and $\mathrm{hull}(I + P_G^{[3]}) = (x^2, xy, y^3)$ is a P-primary component of I.

5.2.2 Regular Sequence Computation for Pseudo-Primary Ideal

We can compute a regular sequence in a P-pseudo-primary ideal I from one of P by the following lemma. Since a generator of P may be more easily than one of I, it tends to be less time-consuming.

Lemma 56
Let I be a P-pseudo-primary ideal and $u = \{f_1, \ldots, f_c\}$ a regular sequence in P. Then, for efficiently large integers $m_1, \ldots, m_c$, $\{f_1^{m_1}, \ldots, f_c^{m_c}\}$ is a regular sequence in I.

Proof By Theorem 26 in [9], $\{f_1^{m_1}, \ldots, f_c^{m_c}\}$ is a regular sequence for any positive integers $m_1, \ldots, m_c$. Since I is P-pseudo-primary, it follows that $\sqrt{I} = P$. Thus, for efficiently large integer $m_1, \ldots, m_c$, $\{f_1^{m_1}, \ldots, f_c^{m_c}\} \subset I$ and it is a regular sequence in I. ∎

Since $\sqrt{(I : (I : P^\infty)^\infty)} = P$ if P is isolated, we obtain the following Corollary. From $\mathrm{codim}(P) = \mathrm{codim}((I : (I : P^\infty)^\infty))$ and Lemma 12, we can compute the equidimensional hull $\mathrm{hull}((I : (I : P^\infty)^\infty))$ by using a regular sequence in P.

Corollary 57
Let I be an ideal and P its isolated prime divisor. Let $u = \{f_1, \dots, f_c\}$ be a regular sequence in P. Then, for efficiently large integer m, $\{f_1^m, \dots, f_c^m\}$ is a regular sequence in $(I : (I : P^\infty)^\infty)$.

5.2.3 Equidimensional Hull Computation with MIS

Next, we devise another computation of $\mathrm{hull}(I + P^m)$ based on *maximal independent set* (MIS) which tends to be much efficient than computations based on Proposition 12. Similarly, by this technique we can replace I with $IK[X]_{K[U]^\times} \cap K[X]$ at the first step of LPA.

Lemma 58 ([5], Lemma 39)
Let I be a P-hull-primary ideal. For a maximal independent set U of P, $\mathrm{hull}(I) = IK[X]_{K[U]^\times} \cap K[X]$.

Proof Let $\mathcal{Q}$ be a primary decomposition of I. Then, $\mathrm{hull}(I)$ is the unique primary component disjoint from $K[U]^\times$. Thus, $IK[X]_{K[U]^\times} \cap K[X] = \bigcap_{Q \in \mathcal{Q}, Q \cap K[U]^\times = \emptyset} Q = \mathrm{hull}(I)$. ∎

Example 59
For $I = (x) \cap (x^2, y)$ and $P = (x)$ in $\mathbb{Q}[X] = \mathbb{Q}[x, y]$, we obtain $U = \{y\}$ is a maximal independent set of P. Then, $\mathrm{hull}(I) = (x) = I\mathbb{Q}[X]_{\mathbb{Q}[U]^\times} \cap \mathbb{Q}[X]$.

6 Further Discussion of Local Primary Algorithm

In this section, we devise another algorithm "LPA-($P_G^{[m]}$+MIS) without DIQ" to compute the particular primary component, without double ideal quotient and its variants. The algorithm uses equidimensional hull to generate primary component in the similar way as LPA. As different points, it uses maximal independent set for another criterion of prime divisor and generalized splitting tool for an additional criterion of primary component.

First, we introduce a new criterion for prime divisors using maximal independent set instead of double ideal quotient.

Proposition 60 (Criterion 8)
Let I be an ideal and P a prime ideal in $K[X]$. Then the following statements are equivalent.

1. $P \in \mathrm{Ass}(I)$.
2. $(I' : P^\infty) \neq I'$, *where $I' = IK[X]_{K[U]^\times} \cap K[X]$ for a maximal independent set U of P.*

Proof Let $\mathcal{Q}$ be a primary decomposition of I. To prove that (1) implies (2), we remark that $P \in \mathrm{Ass}(I)$ leads $P \in \mathrm{Ass}(I')$ from Lemma 72 and $P \cap K[U]^\times = \emptyset$. Thus, we obtain that $(I' : P^\infty) \neq I'$ since $P \notin \mathrm{Ass}((I' : P^\infty))$. Next, we show (2) implies (1). Since $(I' : P^\infty) \neq I'$, there is a prime divisor $P' \in \mathrm{Ass}(I')$ containing P. Then $P' \cap K[U]^\times = \emptyset$ and $\dim(P') \leq \dim(P) = \#U$. From Lemma 72, $P' \in \mathrm{Ass}(I)$ and thus $\dim(P') \geq \#U$. Hence, $\dim(P) = \dim(P')$ and $P = P' \in \mathrm{Ass}(I)$. ∎

Example 61
Let $I = (x^2) \cap (x^3, y)$ and $P = (x)$ in $\mathbb{Q}[X] = \mathbb{Q}[x, y]$. Then, $U = \{y\}$ is the maximal independent set of P and $I' = I\mathbb{Q}[X]_{\mathbb{Q}[U]^\times} \cap \mathbb{Q}[X] = (x^2)$. Since $(I' : P^\infty) = \mathbb{Q}[X] \neq I'$, we get $P \in$ $\mathrm{Ass}(I)$.

Next, we introduce a *P-pseudo-descending chain* to devise a generalized splitting tool and a new criterion for isolated prime divisors. It is a generalization of P^m and $P_G^{[m]}$ in [5].

Definition 62 (P-pseudo-descending chain)
Let P be a prime ideal and $J_1 \supset J_2 \supset J_3 \supset \cdots$ a descending chain of P-pseudo-primary ideals. We say that $J_1 \supset J_2 \supset J_3 \supset \cdots$ is a P-pseudo-descending chain *if $PJ_m \supset J_{m+1}$ for every positive integer m.*

Example 63
As an easy example, $P \supset P^2 \supset P^3 \supset \cdots$ is a P-pseudo-descending chain. For a generator G of P, $P_G^{[1]} \supset P_G^{[2]} \supset P_G^{[3]} \supset \cdots$ is a P-pseudo-descending chain since $P_G^{[m]}$ is P-pseudo-primary and $PP_G^{[m]} \supset P_G^{[m+1]}$ for every m.

Remark 64
We remark that a P-pseudo-descending chain is not always P-filtration i.e. it does not always satisfy the other inclusion $PJ_m \subset J_{m+1}$.

We can use a P-pseudo-descending chain to generate P-primary component as Lemma 65, a generalization of Proposition 48 and Lemma 54.

Lemma 65
Let I be an ideal, P a prime divisor of I and $J_1 \supset J_2 \supset J_3 \supset \cdots$ be a P-pseudo-descending chain. Then, for an efficiently large integer m, $\mathrm{hull}(I + J_m)$ *is a P-primary component of I. Moreover, if* $\mathrm{hull}(I + J_m)$ *is a P-primary component of I for some m, then* $\mathrm{hull}(I + J_{m+1})$ *is also a P-primary component of I.*

Proof Let Q be a P-primary component of I. Since $K[X]$ is Noetherian, there is an efficiently large integer m s.t. $P^m \subset Q$. As $P^m \supset P^{m-1}J_1 \supset P^{m-2}J_2 \supset \cdots \supset PJ_{m-1} \supset J_m$, it follows that $I \subset I + J_m \subset Q$. Here, $\sqrt{I + J_m} = \sqrt{\sqrt{I} + P} = P$ and thus $I + J_m$ is P-pseudo-primary, in particular, P-hull-primary. From Lemma 52, we obtain $\mathrm{hull}(I + J_m)$ is a P-primary component of I. Next, we show the second statement. If $\mathrm{hull}(I + J_m)$ is a P-primary component of I for some m, then it follows that $I \subset I + J_{m+1} \subset I + J_m \subset \mathrm{hull}(I + J_m)$. Thus, $\mathrm{hull}(I + J_{m+1})$ is a P-primary component of I from Lemma 52. ∎

Example 66
Let $I = (x^2, xy)$, $P = (x, y)$ and $J_m = (x^m, y^m)$. We obtain $\mathrm{hull}(I + J_m) = (x^2, xy, y^m)$ *is a P-primary component if $m \geq 2$.*

Here, we devise a generalized splitting tool and find an integer m s.t. $\mathrm{hull}(I + J_m)$ is a P-primary component as follows.

Proposition 67 (Generalized Splitting Tool)
Let I be an ideal, P a prime divisor of I and $J_1 \supset J_2 \supset J_3 \supset \cdots$ be a P-pseudo-descending chain. Then, for an efficiently large integer m,

$$I = (I : P^\infty) \cap (I + J_m).$$

In particular, for such m, $\mathrm{hull}(I + J_m)$ *is a P-primary component of I.*

Proof By Lemma 83, $I = (I : P^\infty) \cap (I + P^m)$ for an efficiently large integer m. As $J_m \subset P^m$, it follows that

$$I = (I : P^\infty) \cap (I + P^m) \supset (I : P^\infty) \cap (I + J_m) \supset I$$

and thus $I = (I : P^\infty) \cap (I + J_m)$. Since $(I : P^\infty)$ does not have P-primary component and $I + J_m$ is P-hull-primary, we obtain $\mathrm{hull}(I + J_m)$ is a P-primary component of I. ∎

Example 68
Let $I = (x^2, xy)$, $P = (x, y)$ and $J_m = (x^m, y^m)$. We obtain $I = (I : P^\infty) \cap (I + J_2) = (x) \cap (x^2, xy, y^2)$ and (x^2, xy, y^2) is a P-primary component of I.

A P-pseudo-descending chain gives us the following criteria for isolated prime divisors.

Theorem 69 (Criterion 9)
Let I be an ideal, P a prime divisor of I and $J_1 \supset J_2 \supset J_3 \supset \cdots$ a P-pseudo-descending chain. We suppose $\mathrm{hull}(I + J_m)$ is a P-primary component of I for some m. Then, the following statements are equivalent.

1. *P is an isolated prime divisor of I.*
2. $\mathrm{hull}(I + J_m) = \mathrm{hull}(I + J_{m+1})$.

Proof First, we show (1) implies (2). By Lemma 65, $\mathrm{hull}(I + J_{m+1})$ is also a P-primary component of I. Since P is isolated, the P-primary component is unique and $\mathrm{hull}(I + J_m) = \mathrm{hull}(I + J_{m+1})$. Second, we show (2) implies (1). Let $R = K[X]_P/IK[X]_P$. Since $I + J_m$ is P-hull-primary, it follows that $\mathrm{hull}(I + J_m) = (I + J_m)K[X]_P \cap K[X]$ and thus $\mathrm{hull}(I + J_m)R = J_mR$. As $\mathrm{hull}(I + J_m) = \mathrm{hull}(I + J_{m+1})$, we get $J_mR = J_{m+1}R$. Thus, $J_m \supset PJ_m \supset J_{m+1}$ and it follows that $J_mR \supset PJ_mR \supset J_{m+1}R = J_mR$, hence, $J_mR = PJ_mR$. Since J_mR is finitely generated $K[X]_P$-module, we obtain $J_mR = 0$ by Nakayama's Lemma. Thus, $J_mK[X]_P = IK[X]_P$ and $P \in \mathrm{Ass}(\sqrt{I})$, otherwise, $IK[X]_P$ has two or more prime divisors. Therefore, P is isolated. ∎

Example 70
Let $I = (x^2) \cap (x^3, y)$. For $P_1 = (x)$, it follows that $\mathrm{hull}(I + P_1^2) = \mathrm{hull}(I + P_1^3) = (x^2)$ is a P_1-primary component. Thus, P_1 is the isolated prime divisor of I. On the other hand, for $P_2 = (x, y)$ and $J_m = (x^m, y^m)$, $\mathrm{hull}(I + J_3) = (x^3, x^2y, y^3)$ is a P_2-primary component and hull$(I + J_3) \supsetneq \mathrm{hull}(I + J_4) = (x^3, x^2y, y^4)$; thus P_2 is embedded.

Remark 71
An integer m s.t. $\mathrm{hull}(I+J_m)$ is a P-primary component of I may be smaller than m' s.t. $\mathrm{hull}(I+P^{m'})$ is a P-primary component of I. Thus, we may compute a primary component more easily by $\mathrm{hull}(I + P_G^{[m]})$.

Algorithm 2 is another version of Local Primary Algorithm, without using DIQ. As J_m, we use $P_G^{[m]}$ (currently we think this J_m is the best), for efficient computations and maximal independent set in steps of the following algorithm.

7 Experiments and Observations

We made an implementation on the computer algebra system Risa/Asir [11] and apply it to several examples as experiments. We revisited old examples in [5], $I_1(n)$ and $A_{k,m,n}$. The former $I_1(n) = (x^2) \cap (x^4, y) \cap (x^3, y^3, (z + 1)^n + 1)$ is an ideal whose embedded primary components are hard to compute. If n is considerable large, it is difficult to compute a full primary decomposition of $I_1(n)$ though the isolated divisor $P_1 = (x)$ can be detected pretty easily. The latter $A_{k,m,n}$ defined in [13] is more valuable for mathematics and its primary decomposition has important meanings in Computer Algebra for Statistics. We newly considered $T_1, \ldots, T_{10}$ that appear in [7] for benchmarks of primary decomposition. We describe the more details of ideals in A.2. Timings are measured on a PC with Intel Core i7-8700B CPU with 32GB memory.

Algorithm 2 Local Primary Algorithm Without Double Ideal Quotient

Input: I: an ideal, P: a prime ideal in $K[X]$
Output: • a P-primary component if P is a prime divisor
• "P is not a prime divisor" otherwise

$U \leftarrow$ a Maximal Independent Set of P, $I' \leftarrow IK[X]_{K[U]^{\times}} \cap K[X]$
$G \leftarrow \{f_1, \ldots, f_s\}$ a generator of P, $m \leftarrow 1$
if $(I' : P^{\infty}) = I'$ **then**
  return "P is not a prime divisor " (**Criterion 8**)
end if
while $(I' : P^{\infty}) \cap (I' + P_G^{[m]}) \neq I'$ **do**
  $m \leftarrow m + 1$ (**Proposition 67**)
end while
$Q_m \leftarrow \mathrm{hull}(I' + P_G^{[m]}) = (I' + P_G^{[m]})K[X]_{K[U]^{\times}} \cap K[X]$ (**Lemma 58**)
$Q_{m+1} \leftarrow \mathrm{hull}(I' + P_G^{[m+1]}) = (I' + P_G^{[m+1]})K[X]_{K[U]^{\times}} \cap K[X]$
if $Q_m = Q_{m+1}$ **then**
  return "Q_m is the isolated P-primary component of I " (**Criterion 9**)
else
  return "Q_m is an embedded P-primary component of I " (**Criterion 9**)
end if

Now, we explain the details of Local Primary Algorithms (LPAs). From Proposition 12, the primitive LPA use *double ideal quotient* and *regular sequence* to compute *equidimensional hull*. To compute a regular sequence in $I + P_G^{[m]}$ and that in $(I : (I : P^{\infty})^{\infty})$ efficiently, we use Lemma 56 and Corollary 57 respectively. As improved versions, LPA-$P_G^{[m]}$ is an implementation based on Lemma 54 and LPA-MIS is one from Lemma 58 and Criteria 3, 4. Both methods are implemented in LPA-($P_G^{[m]}$+MIS). The new algorithm LPA-($P_G^{[m]}$+MIS) without DIQ is based on Algorithm 2. Here, as a reference, we show the timings of a full primary decomposition function `noro_pd.syci_dec` in Table 6.

In all Figures, the horizontal axis shows isolated or embedded prime divisors and the vertical axis represents the timing (in seconds) of each prime divisor. In particular, the embedded prime divisors are in order of decreasing dimension.

7.1 Computation of Isolated Components

First, we apply LPAs to isolated primary components. In Table 1 and Table 2, we can see LPAs have clearly effectiveness by their specialities. We call an algorithm *stable* for an ideal if the *statistical standard deviation* of timing data for their prime divisors is small. Figure 1 and Table 3 show that LPA is stable for T_1 since the the statistical standard deviation is 4.17, which is much smaller than those of LPA-MIS and LPA-($P_G^{[m]}$+MIS). On the other hand, both LPA-MIS and LPA-($P_G^{[m]}$+MIS) without DIQ take much time for some cases and are unstable since the statistical standard deviations are over 100 times of that of LPA. Also, we can see its instability in Figures 2 and 3, where we limit the maximum to 35 seconds. The main reason is that MIS-localization becomes very time-consuming for specific ideals and prime ideals. However, when MIS-localization is efficient, timings of LPA-MIS and LPA-($P_G^{[m]}$+MIS) without DIQ are much faster than those of LPA. There are almost no difference between timings of LPA-MIS and LPA-($P_G^{[m]}$+MIS) without DIQ since MIS-localization is very effective and it can reduce the timings of other parts. As a summary of analysis for isolated examples,

- LPAs have clearly effectiveness by their specialities.
- LPA is stable, on the other hand, both LPA-MIS and LPA-($P_G^{[m]}$+MIS) without DIQ are unstable due to *strange* behavior of MIS-localization. However, it is much useful than LPA when MIS-localization works well.

Ideals\Algorithms	LPA	LPA-MIS	LPA-($P_G^{[m]}$+MIS) w/o DIQ
$I_1(100), P_1$	0.01	0.007	0.006
$I_1(200), P_1$	0.02	0.01	0.01
$I_1(300), P_1$	0.03	0.01	0.01
$I_1(400), P_1$	0.04	0.02	0.01
$I_1(500), P_1$	0.05	0.02	0.02
$A_{3,4,5}, P_2$	14.1	> 7200	> 7200
T_1, P_3	12.3	> 7200	> 7200
T_1, P_4	28.2	0.20	0.19
T_2, P_5	50.0	> 7200	> 7200
T_3, P_6	0.96	0.04	0.04
T_4, P_7	4.11	7.74	7.84
T_5, P_8	5.22	0.07	0.07
T_6, P_9	0.13	0.02	0.01
T_7, P_{10}	25.5	0.21	0.21
T_8, P_{11}	0.06	0.02	0.02
T_9, P_{12}	2.42	1.78	1.73
T_{10}, P_{13}	151	2.81	2.81

Table 1: Local Primary Algorithm (Isolated)

Ideals (# of isolated components)	LPA	LPA-MIS	LPA-($P_G^{[m]}$+MIS) w/o DIQ
T_1 (49)	100	73.4	75.5
T_2 (15)	0	0	0
T_3 (47)	97.8	82.9	82.9
T_4 (40)	100	95.0	95.0
T_5 (14)	100	71.4	71.4
T_6 (48)	100	100	100
T_7 (55)	100	89.0	90.9
T_8 (37)	91.8	67.5	67.5
T_9 (15)	100	26.6	40.0
T_{10} (76)	100	43.4	46.0

Table 2: Comparison among LPAs (the ratios of isolated primary components which each LPA could compute more efficiently than the specified full primary decompositions.)

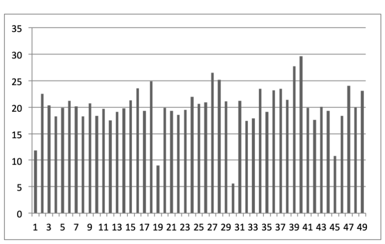

Fig. 1: LPA (49 isolated prime divisors of T_1)

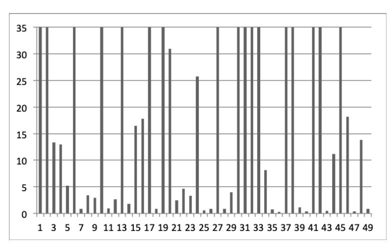

Fig. 2: LPA-MIS (49 isolated prime divisors of T_1)

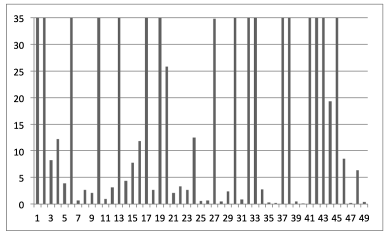

Fig. 3: LPA-($P_G^{[m]}$+MIS) without DIQ (49 isolated prime divisors of T_1)

Ideals \ Algorithms	LPA	LPA-MIS (LPA-MIS/LPA)	LPA-($P_G^{[m]}$+MIS) w/o DIQ (LPA/(LPA-($P_G^{[m]}$+MIS) w/o DIQ))
T_1	4.17	457 (109)	478 (114)
T_3	173	428 (2.47)	428 (2.47)
T_4	0.68	14.9(21.9)	14.8 (21.7)
T_5	2.65	541(204)	541 (204)
T_7	4.26	282(66.1)	281 (65.9)
T_8	327	438(1.33)	439 (1.34)
T_9	0.11	582 (5290)	584 (5309)
T_{10}	16.8	557 (33.1)	562 (33.4)

Table 3: The statistical standard deviations of timing data for isolated prime divisors, where we limit the maximum to 1200 seconds

7.2 Computation of Embedded Components

In Table 4, the primitive LPA is not practical for some examples since the cost of computing hull($I + P^m$) is much high. Comparing LPA and LPA-$P_G^{[m]}$ (also LPA-MIS and LPA-($P_G^{[m]}$+MIS)), we can see the technique $P_G^{[m]}$-products is effective for most cases. As algorithms using MIS-localization, LPA-($P_G^{[m]}$+MIS) and LPA-($P_G^{[m]}$+MIS) without DIQ have good effectivenesses by their specialities for many cases, for examples, $(I_1(n), P_{14})$, $(A_{2,4,4}, P_{15})$, $(A_{2,3,7}, P_{16})$, (T_1, P_{17}), (T_4, P_{21}), (T_7, P_{24}), (T_8, P_{25}), (T_{10}, P_{27}) and so on. From Table 4, we can see MIS-technique is efficient for many cases. However, there are some examples s.t. MIS-localization is not efficient, for instance, (T_1, P_{18}) and (T_3, P_{20}). As a consideration of the ration of such non-efficient case, in Table 5, we can see both LPA-($P_G^{[m]}$+MIS) and LPA-($P_G^{[m]}$+MIS) are effective for 96.6% of embedded prime divisors of T_1 i.e. MIS-localization is efficient for *most* embedded prime divisors of T_1. In Figures 4,5 and 7, we can see LPAs using MIS are unstable due to MIS-localization, comparing LPA-$P_G^{[m]}$. Same as isolated components, there are almost no difference between timings of LPA-($P_G^{[m]}$+MIS) and LPA-($P_G^{[m]}$+MIS) without DIQ since MIS-localization is much powerful and we can ignore the timings for computation of DIQ. In summary,

- The technique $P_G^{[m]}$-products is effective for most cases.
- Both LPA-($P_G^{[m]}$+MIS) and LPA-($P_G^{[m]}$+MIS) without DIQ are much efficient to compute specific embedded components for most prime divisors.
- MIS-localization is very powerful but unstable, compared to LPA-$P_G^{[m]}$.

Ideals \ Algorithms	LPA	LPA-$P_G^{[m]}$	LPA-MIS	LPA-($P_G^{[m]}$+MIS)	LPA-($P_G^{[m]}$+MIS) w/o DIQ
$I_1(100), P_{14}$	0.09	0.07	0.01	0.01	0.007
$I_1(200), P_{14}$	0.17	0.14	0.02	0.02	0.01
$I_1(300), P_{14}$	0.29	0.25	0.02	0.02	0.01
$I_1(400), P_{14}$	0.41	0.31	0.03	0.03	0.02
$I_1(500), P_{14}$	0.43	0.38	0.03	0.02	0.03
$A_{2,4,4}, P_{15}$	1707	5.50	0.56	0.25	0.32
$A_{2,3,7}, P_{16}$	143	25.1	0.60	0.37	0.41
T_1, P_{17}	73.8	71.8	0.27	0.22	0.20
T_1, P_{18}	61.6	58.2	>7200	>7200	>7200
T_2, P_{19}	214	188	>7200	>7200	>7200
T_3, P_{20}	0.75	0.76	29.6	29.5	29.5
T_4, P_{21}	10.9	9.53	0.12	0.10	0.08
T_5, P_{22}	>7200	63.0	>7200	2.82	1.13
T_6, P_{23}	>7200	5.83	>7200	0.13	0.05
T_7, P_{24}	86.3	41.5	5.89	0.21	0.19
T_8, P_{25}	3.32	0.27	0.08	0.04	0.02
T_9, P_{26}	9.54	8.18	>7200	>7200	>7200
T_{10}, P_{27}	4338	256	668	0.89	0.80

Table 4: Local Primary Algorithm (Embedded)

Ideals (# of embedded components)	LPA-$P_G^{[m]}$	LPA-($P_G^{[m]}$+MIS)	LPA-($P_G^{[m]}$+MIS) w/o DIQ
T_1 (120)	41.6	96.6	96.6

Table 5: Comparison of LPAs (the ratios of embedded primary components which each LPA could compute more efficiently than the specified full primary decomposition of T_1)

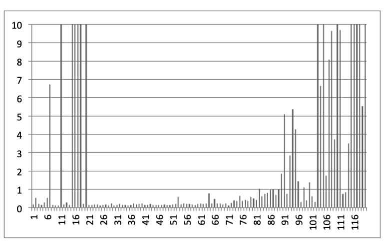

Fig. 4: LPA-($P_G^{[m]}$+MIS)
(120 embedded prime divisors of T_1)
upper limit: 10 seconds

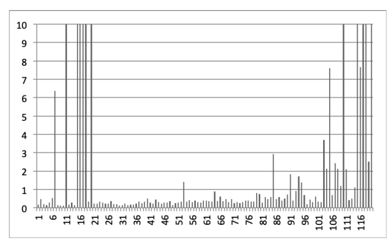

Fig. 5: LPA-($P_G^{[m]}$+MIS) w/o DIQ
(120 embedded prime divisors of T_1)
upper limit: 10 seconds

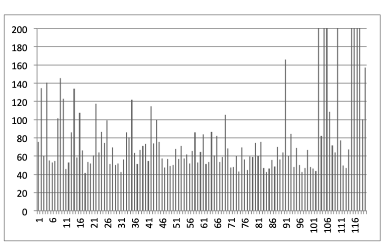

Fig. 6: LPA-$P_G^{[m]}$
(120 embedded prime divisors of T_1)
upper limit: 200 seconds

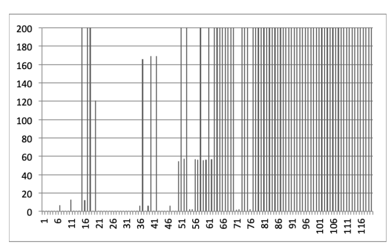

Fig. 7: LPA-MIS
(120 embedded prime divisors of T_1)
upper limit: 200 seconds

Ideals \ Algorithms	full primary decomposition (`noro_pd.syci_dec`)
$I_1(100)$	0.28
$I_1(200)$	11.3
$I_1(300)$	66.7
$I_1(400)$	167
$I_1(500)$	73.3
$A_{2,4,4}$	3.42
$A_{2,3,7}$	31.2
$A_{3,4,5}$	> 7200
T_1	62.7
T_2	30.0
T_3	63.9
T_4	35.0
T_5	49.8
T_6	5.58
T_7	261
T_8	1.82
T_9	5.24
T_{10}	324

Table 6: The timings of full primary decompositions (Reference)

7.3 Summary on Computational behavior

In isolated cases, LPAs have clearly effectiveness by their specialities. In embedded cases, the technique P_G^m-products is a useful way. For both cases, MIS-localization is very efficient for many ideals and prime divisors, however, it is unstable. To make our LPAs more effective, we need improvements of DIQ or MIS-localization. Since methods without MIS (LPA and LPA-$P_G^{[m]}$) are stable, improvements of DIQ gives us stable LPA-algorithms. On the other hand, if we succeed improvements of MIS-localization for every cases, we also have efficient algorithms.

8 Conclusion and Future Work

In commutative algebra and algebraic geometry, the operation of "localization by a prime ideal" is widely known as a basic tool. In the paper, we focus on computing a primary component from only its prime divisor and propose a new effective localization Local Primary Algorithm (LPA). It mainly uses double ideal quotient (DIQ) (and its variants), and localization by maximal independent set (MIS). As an enhanced full paper version of [5], this paper contains detailed proofs, additional examples and new algorithms. Moreover, we took benchmarks for many examples to examine the effectiveness of LPA coming from its speciality. In the additional discussion, we invent another algorithm using a well-known splitting tool and maximal independent set instead of DIQ to compare it and the original LPAs. From experiments, we can see MIS-localization is very effective for many cases, however, it is *unstable* and there are some examples which are very time-consuming. We conclude that effectiveness of the LPAs depends on ideals and it would be better, at the moment, to apply them in parallel.

In future work, to make our LPAs very practical we shall continue to improve it through obtaining timing data for a lot of larger examples. In particular, we need to invent effective algorithms to compute double ideal quotient and MIS-localization. To solve it, we can apply so-called *modular techniques* using computations over finite fields for those over the rational field by Chinese Remainder Theorem and rational reconstruction. Since intermediate coefficient growth does not happen over a finite field, it is expected to reduce time of computation over the rational field dramatically. The first author just reported his first attempt of such modular techniques in the recent paper ([4]).

Another work shall be to apply our primary component criteria to *probabilistic or inexact* methods for primary decomposition, such as numerical ones. Probabilistic or inexact ways may have low computational costs but low accuracy for outputs. Hence, our criteria using double ideal quotient can guarantee their outputs. For example, we are thinking to combine our LPAs and Numerical Primary Decomposition in [8] to compute possible prime divisors and primary components.

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A Fundamental Lemmas and their Proofs (Appendix)

A.1 Lemmas and Definitions

The following lemma is an easy but fundamental criterion for primary component using localization.

Lemma 72 ([5], Lemma 4)
Let I be an ideal and P its prime divisor. If S is a multiplicatively closed set with $P \cap S = \emptyset$ and Q is a P-primary ideal, then the following conditions are equivalent.
(A) *Q is a primary component of I*
(B) *Q is a primary component of $IK[X]_S \cap K[X]$*

Proof First, (A) implies (B) from Proposition 4.9 in [1] . For primary decompositions $\mathcal{Q}$ of I and $\mathcal{Q}'$ of $IK[X]_S \cap K[X]$ with $Q \in \mathcal{Q}'$, we obtain $\{Q' \in \mathcal{Q} \mid Q' \cap S \neq \emptyset\} \cup \mathcal{Q}'$ is also a primary decomposition of I. Hence, (B) implies (A). ∎

In particular, one or more isolated primary components of I are isolated in $IK[X]_S \cap K[X]$ if the localization is not trivial.

Example 73
For $I = (x^2, xy) \subset K[X] = K[x, y]$, we obtain that (x) is the isolated primary component of both I and $IK[X]_{(x)} \cap K[X] = (x)$.

We define a special subset of $\mathrm{Ass}(I)$, which has a good relationship to localization. The localization by an isolated set can be expressed as intersection of primary components whose prime divisors are in the isolated set.

Definition 74 ([1], Chapter 4)
Let I be an ideal. A subset $\mathcal{P}$ of $\mathrm{Ass}(I)$ is said to be isolated *if it satisfies the following condition: for a prime divisor $P' \in \mathrm{Ass}(I)$, if $P' \subset P$ for some $P \in \mathcal{P}$, then $P' \in \mathcal{P}$.*

Lemma 75 ([1], Theorem 4.10)
Let I be an ideal and $\mathcal{P}$ an isolated set contained in $\mathrm{Ass}(I)$. For a multiplicatively closed set $S = K[X] \setminus \bigcup_{P \in \mathcal{P}} P$ and a primary decomposition $\mathcal{Q}$ of I, $IK[X]_S \cap K[X] = \bigcap_{Q \in \mathcal{Q},\, \sqrt{Q} \in \mathcal{P}} Q$.

Example 76
For $I = (x^2(x + 1), x(x + 1)y) \subset K[X] = K[x, y]$, $\mathcal{P} = \{(x), (x, y)\}$ is an isolated subset of $\mathrm{Ass}(I) = \{(x), (x + 1), (x, y)\}$. Let $S = K[X] \setminus \bigcup_{P \in \mathcal{P}} P$. Then, $IK[X]_S \cap K[X] = (x) \cap (x^2, y)$.

The following lemma tells us when primary component intersects a multiplicatively closed set. It is used to prove Lemma 29, a criterion for localization.

Lemma 77 ([5], Lemma 7)
Let $\mathcal{Q}$ be a primary decomposition of I and $Q \in \mathcal{Q}$. For a multiplicatively closed set S, the following conditions are equivalent.
(A) $IK[X]_S \cap K[X] \subset IK[X]_{\sqrt{Q}} \cap K[X]$.
(B) $Q \cap S = \emptyset$.

Proof Show (A) implies (B). As $IK[X]_{\sqrt{Q}} \cap K[X] \subset Q$, $IK[X]_S \cap K[X] = \bigcap_{Q' \in \mathcal{Q}, Q' \cap S = \emptyset} Q' \subset Q$. Since $\mathcal{Q}$ is irredundant, $IK[X]_S \cap K[X]$ has $\sqrt{Q}$-primary component. Thus, $Q \cap S = \emptyset$. Now, we show (B) implies (A). Then, $\sqrt{Q} \cap S = \emptyset$ and $Q' \cap S = \emptyset$ for any $Q' \in \mathcal{Q}$ s.t. $Q' \subset \sqrt{Q}$. Thus, $IK[X]_{\sqrt{Q}} \cap K[X] = \bigcap_{Q' \subset \sqrt{Q}} Q'$ implies $IK[X]_S \cap K[X] \subset IK[X]_{\sqrt{Q}} \cap K[X]$. ∎

Example 78
For $I = (x) \cap (x+1) \cap (x^2, y) \subset \mathbb{Q}[X] = \mathbb{Q}[x, y]$, let $S = \mathbb{Q}[X] \setminus (x, y)$. Then, $I\mathbb{Q}[X]_S \cap \mathbb{Q}[X] \subset I\mathbb{Q}[X]_{\sqrt{(x)}} \cap \mathbb{Q}[X]$ and $(x) \cap S = \emptyset$. On the other hand, $I\mathbb{Q}[X]_S \cap \mathbb{Q}[X] \not\subset I\mathbb{Q}[X]_{\sqrt{(x+1)}} \cap \mathbb{Q}[X]$ and $(x+1) \cap S \neq \emptyset$.

The following lemma tells that primary ideal has a similar property to one of prime ideal.

Lemma 79 ([5], Lemma 16)
Let I and J be ideals. Let Q be a primary ideal. If $IJ \subset Q$ and $J \not\subset \sqrt{Q}$, then $I \subset Q$. In particular, if $I \cap J \subset Q$ and $J \not\subset \sqrt{Q}$, then $I \subset Q$.

Proof Let $f \in I$ and $g \in J \setminus \sqrt{Q}$. Since Q is $\sqrt{Q}$-primary, $fg \in IJ \subset Q$ implies $f \in Q$. ∎

Example 80
Let $I = (x)$, $J = (x+1)$ and $Q = (x, y^2)$. Then, $I \cap J \subset (x(x+1)) \subset (x, y^2) = Q$ and $J = (x+1) \not\subset \sqrt{Q} = (x, y)$. Thus, $I = (x) \subset Q = (x, y^2)$.

Hull-primary ideal has a similar property to one of primary ideal as follows.

Lemma 81 ([5], Lemma 17)
Let I be a P-hull-primary and Q a P-primary ideal. If $I \subset Q$, then $\mathrm{hull}(I) \subset Q$.

Proof Let $\mathcal{Q}$ be a primary decomposition of I and $J = \bigcap_{Q' \in \mathcal{Q}, Q' \neq \mathrm{hull}(I)} Q'$. Then $I = \mathrm{hull}(I) \cap J \subset Q$ and $J \not\subset P$. Since Q is P-primary, we obtain $\mathrm{hull}(I) \subset Q$ by Lemma 79. ∎

Example 82
Let $I = (x^2) \cap (x^3, y) \cap (x+1, y+1)$ and $Q = (x)$. Then, $I \subset Q$ and $\mathrm{hull}(Q) = (x^2) \subset Q$.

Next, we remark the "splitting tool", one of the most important tool for primary decomposition.

Lemma 83 ([14], Proposition 3.53)
Let I and J be ideals. Then, for a sufficiently large integer m,

$$I = (I : J^\infty) \cap (I + J^m).$$

Example 84
For $I = (x^2, xy)$ and $J = (x, y)$,

$$I = (I : J^\infty) \cap (I + J^2) = (x) \cap (x^2, xy, y^2).$$

Also, we recall the famous Prime Avoidance Lemma.

Lemma 85 ([1], Proposition 1.11)
(i) *Let* $P_1, \ldots, P_m$ *be prime ideals and let* I *be an ideal contained in* $\bigcup_{i=1}^m P_i$. *Then,* $I \subset P_i$ *for some* i.
(ii) *Let* $I_1, \ldots, I_m$ *be ideals and let* P *be a prime ideal containing* $\bigcap_{i=1}^m I_i$. *Then* $P \supset I_i$ *for some* i. *If* $P = \bigcap_{i=1}^m I_i$, *then* $P = I_i$ *for some* i.

Finally, We add a proof of Lemma 18 in Sect. 2.2 as follows.

Lemma 18 ([5], Lemma 19)
Let I *and* J *be ideals,* Q *a primary ideal and* $\mathcal{Q}$ *a primary decomposition of* I. *Then,*

$$(Q : J) = \begin{cases} Q & (J \not\subset \sqrt{Q}), \\ K[X] & (J \subset Q), \\ \sqrt{Q}\text{-primary ideal properly containing } Q & (J \not\subset Q, J \subset \sqrt{Q}), \end{cases} \tag{1}$$

$$(Q : J^\infty) = \begin{cases} Q & (J \not\subset \sqrt{Q}), \\ K[X] & (J \subset \sqrt{Q}), \end{cases} \tag{2}$$

$$(I : J) = \bigcap_{Q \in \mathcal{Q}, J \not\subset \sqrt{Q}} Q \cap \bigcap_{Q \in \mathcal{Q}, J \not\subset Q, J \subset \sqrt{Q}} (Q : J), \tag{3}$$

$$(I : J^\infty) = (I : \sqrt{J}^\infty) = \bigcap_{Q \in \mathcal{Q}, J \not\subset \sqrt{Q}} Q. \tag{4}$$

Proof First, (1) can be proved directly from a remark before Proposition 3.56 in [14]. Second, we show (2). We note that $J \not\subset \sqrt{Q}$ implies $J^m \not\subset \sqrt{Q}$ for any positive integer m, and thus $(Q : J^m) = Q$ from (1). Since $K[X]$ is Noetherian, $(Q : J^\infty) = (Q : J^m)$ for a sufficiently large m. Thus, we obtain $(Q : J^\infty) = Q$ if $J \not\subset \sqrt{Q}$. If $J \subset \sqrt{Q}$, then $J^m \subset Q$ for a sufficiently large m and $(Q : J^\infty) = (Q : J^m) = K[X]$ from (1). Third, we prove (3). From $I = \bigcap_{Q \in \mathcal{Q}} Q$ and (1), we obtain

$$\begin{aligned} (I : J) &= (\bigcap_{Q \in \mathcal{Q}} Q : J) = \bigcap_{Q \in \mathcal{Q}} (Q : J) \\ &= \bigcap_{Q \in \mathcal{Q}, J \not\subset \sqrt{Q}} (Q : J) \cap \bigcap_{Q \in \mathcal{Q}, J \not\subset Q, J \subset \sqrt{Q}} (Q : J) \cap \bigcap_{Q \in \mathcal{Q}, J \subset Q} (Q : J) \\ &= \bigcap_{Q \in \mathcal{Q}, J \not\subset \sqrt{Q}} Q \cap \bigcap_{Q \in \mathcal{Q}, J \not\subset Q, J \subset \sqrt{Q}} (Q : J) \cap K[X] \\ &= \bigcap_{Q \in \mathcal{Q}, J \not\subset \sqrt{Q}} Q \cap \bigcap_{Q \in \mathcal{Q}, J \not\subset Q, J \subset \sqrt{Q}} (Q : J). \end{aligned}$$

Finally, we show (4). From $I = \bigcap_{Q \in \mathcal{Q}} Q$ and (1), we obtain

$$\begin{aligned} (I : J^\infty) &= \bigcap_{Q \in \mathcal{Q}, J \not\subset \sqrt{Q}} (Q : J^\infty) \cap \bigcap_{Q \in \mathcal{Q}, J \subset \sqrt{Q}} (Q : J^\infty) \\ &= \bigcap_{Q \in \mathcal{Q}, J \not\subset \sqrt{Q}} Q \cap K[X] = \bigcap_{Q \in \mathcal{Q}, J \not\subset \sqrt{Q}} Q. \end{aligned}$$

Since $J \subset \sqrt{Q}$ is equivalent to $\sqrt{J} \subset \sqrt{Q}$, we obtain $(I : J^\infty) = (I : \sqrt{J}^\infty)$. ∎

A.2 Ideals and Prime Ideals in Experiments

$$
\begin{aligned}
I_1(n) =& (x^2) \cap (x^4, y) \cap (x^3, y^3, (z+1)^n + 1) \subset \mathbb{Q}[x, y, z].\\
A_{3,4,5} =& ((x_{12}x_{23} - x_{13}x_{22})x_{31} - x_{11}x_{32}x_{23} + x_{11}x_{33}x_{22} + (x_{13}x_{32} - x_{12}x_{33})x_{21},\\
& (x_{13}x_{32} - x_{12}x_{33})x_{24} + (-x_{14}x_{32} + x_{12}x_{34})x_{23} + (x_{14}x_{33} - x_{13}x_{34})x_{22},\\
& (x_{14}x_{33} - x_{13}x_{34})x_{25} + (-x_{15}x_{33} + x_{35}x_{13})x_{24} + (x_{15}x_{34} - x_{35}x_{14})x_{23},\\
& (x_{42}x_{23} - x_{43}x_{22})x_{31} - x_{41}x_{32}x_{23} + x_{41}x_{33}x_{22} + (x_{43}x_{32} - x_{42}x_{33})x_{21},\\
& (x_{43}x_{32} - x_{42}x_{33})x_{24} + (-x_{44}x_{32} + x_{42}x_{34})x_{23} + (x_{44}x_{33} - x_{43}x_{34})x_{22},\\
& (x_{44}x_{33} - x_{43}x_{34})x_{25} + (-x_{45}x_{33} + x_{35}x_{43})x_{24} + (x_{45}x_{34} - x_{35}x_{44})x_{23})\\
& \subset \mathbb{Q}[x_{ij} \mid 1 \le i \le 4, 1 \le j \le 5].\\
A_{2,4,4} =& (-x_{21}x_{12} + x_{22}x_{11}, -x_{22}x_{13} + x_{23}x_{12}, -x_{23}x_{14} + x_{24}x_{13}, x_{32}x_{21} - x_{31}x_{22},\\
& x_{33}x_{22} - x_{32}x_{23}, x_{34}x_{23} - x_{24}x_{33}, x_{42}x_{31} - x_{41}x_{32}, x_{43}x_{32} - x_{42}x_{33},\\
& x_{44}x_{33} - x_{43}x_{34}) \subset \mathbb{Q}[x_{ij} \mid 1 \le i \le 4, 1 \le j \le 4].\\
A_{2,3,7} =& (-x_{21}x_{12} + x_{22}x_{11}, -x_{22}x_{13} + x_{23}x_{12}, -x_{23}x_{14} + x_{24}x_{13}, -x_{24}x_{15} + x_{25}x_{14},\\
& - x_{25}x_{16} + x_{26}x_{15}, -x_{26}x_{17} + x_{27}x_{16}, x_{32}x_{21} - x_{31}x_{22}, x_{33}x_{22} - x_{32}x_{23},\\
& x_{34}x_{23} - x_{24}x_{33}, x_{35}x_{24} - x_{25}x_{34}, x_{36}x_{25} - x_{26}x_{35}, x_{37}x_{26} - x_{36}x_{27})\\
& \subset \mathbb{Q}[x_{ij} \mid 1 \le i \le 3, 1 \le j \le 7].\\
T_1 =& (cdefghiz + cdefhjz + bcdeijz, 3cdfghz^3 + 4bdefghj + 4bdehjz^2,\\
& 2bfghijz + fhjz^3, 4bcefhz + cfgijz, cdjz, 3egjz^4 + bcdgij + 2cdhjz^2,\\
& 3defiz + 2defz^2 + 4bcei, 4bcefiz + 3dfhjz^2, cefhjz + bcfiz^2 + giz^4,\\
& 4ceghiz + bcejz) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, z].\\
T_2 =& (3bcegz^2 + 4bcghi + 2bcez^2, bcez + 3dhi, cfgiz^3 + bcdegh, cfgz^4 +\\
& 3cdefgh, 2bcfgiz^2 + bcdegh + z^6, bchz + 4bcg, 4bcdgiz + 2cfhiz^2 +\\
& 3bdfhi, bdefhz + bz^4, 3befgiz + 2cefgz^2 + 4cfhz^2, 3bfh + 4fhi + bz^2)\\
& \subset \mathbb{Q}[b, c, d, e, f, g, h, i, z].\\
T_3 =& (4befjkmz^3 + 2bcdhijlm + cdegkmz^2, cdeghjlz, 2defghilz + 4jlz^6 +\\
& defjlz^2, beghjlmz + 4ceghiz^2 + bdeflz^2) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, k, l, m, z].\\
T_4 =& (2cfhiz^2 + bdefh, bcfijz + 4bcghi, 2cdejz + 4cdfj + ijz^2, bcdfgijz +\\
& cdijz^3, 3bceijz + 3cgijz^2 + beiz^3, 4bchjz + cgiz^2, behj, 3cdefhiz +\\
& 2bdfgjz + 2bchjz^2) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, z].\\
T_5 =& (4bc^2d^2e^2gh^2iz^2 + b^2ciz^9 + 2bceg^2hz^5, bcd^2e^2g^2h^2, bcfhz^5 + b^2dfg^2iz,\\
& 4bc^2e^2f^2h^2i^2z^2 + b^2c^2e^2fh^2iz^3, 2b^2de^2f^2hi^2z + 3b^2c^2e^2h^2i^2)\\
& \subset \mathbb{Q}[b, c, d, e, f, g, h, i, z].\\
T_6 =& (4bcdfghlz + 3bcfhlz^3, befhkl + defghz, 3bdefhijklz + 2cfhjkz^5 +\\
& bdehkz^4, 4befijkl + dgklz^3, bcdefghj + 2bcdegijz + 2bcdhjklz,\\
& cdegijz + 3bcdefk + 4fhklz^2, 2bdeghjkz + cdez^5 + 3eghjz^3,\\
& bcdghijz + cdfhklz + 2bcdhkz^2, 2bcdefi + bhijkl, eghjkz^5 +\\
& 2bcefghjkl, gilz^2 + 2beil, g, 3cdefijkl + 4bcdgjz^3, cdehijz + 4cegjz^3,\\
& bchkl, cdfghklz + befhilz + cdfgjlz, fiz^5 + 2cdfghk + bdfhiz,\\
& befijklz^2 + 3bcdghijl, 2bgijklz + 2bcghil + cefhjz, 2defghjz +
\end{aligned}
$$

$$
\begin{aligned}
&3cefhijz + 3bdghiz) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,k,l,z].\\
T_7 =&(cfghijklz + cdz^7, 3bdikz^7 + 3bcdefghikl + 4bfghkz^5, 3befghijkz+\\
&2bcegijz^3, 3cfhjlz + dfhjlz + 4bdfkl, 3bejz^4 + bdfgjk + 2begjz^2,\\
&cdefgjkz + 3efgjlz^2 + 4elz^5, bcdefghjk, 4cehjlz^4 + 3ceghijkl,\\
&efghjklz, ik, 4beghijkz^3 + 3bdeghijkl, cdefkl + dgjklz, 2bghijlz+\\
&bcdgiz + 4eghjkz, bcehijklz + cdghijlz^2, 2bcdefglz + 2cfgijlz^2+\\
&chz^6, 4bdefhjlz + bdhijlz + 2defgklz, 2cdgiklz + cehklz^2 + 4cghilz,\\
&chjkl, 2bcdhijlz + cgijz^4, bdfhijkz + 4bdijkz^3 + 2dhlz^4)\\
&\subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,k,l,z].\\
T_8 =&(3bejz^4 + bdfgjk + 2begjz^2, cdefgjkz + 3efgjlz^2 + 4elz^5, bcdefghjk,\\
&4cehjlz^4 + 3ceghijkl) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,k,l,z]\\
T_9 =&(3hz^4 + 2cdfg, bdefgh + cfgz^3 + cgz^4, bcgz^2 + cdef + defz,, 3efgh+\\
&bcez + 2bfz^2, 3defh + 2cegh, dehz + 4cgz^2, 2cdefhz + chz^3, 3cdefhz+\\
&2cfghz, 3dfghz + 2efhz^2 + 2bcgz, bdhz + 2efz + 2bhz)\\
&\subset \mathbb{Q}[b,c,d,e,f,g,h,z].\\
T_{10} =&(4cdfhjkz + 4efhijz^2 + cehiz^2, bcdfiz, 3bdefhj + 4cdeghz, cdegkz+\\
&bdiz^3, bcdkz^2 + 2begjk, 2cdefhijz + 3cehijz^3 + bcdhz^4, efhjkz + 3bcfhz,\\
&2bcegiz + 3dghijz + 3fghiz, bdfjz + dfjkz, 4efhikz + 3befhi + 2dfghi,\\
&cdhijz + 2efgkz^2, bcdgikz^2 + bcdfgik, dfgikz, 2bcdghiz + bcegiz^2+\\
&bdfijk, cdefghijz, bcdegijkz + cdefkz^4, 4bdfghjz + bdgkz^3 + 2bcdeij,\\
&cefghijkz + 4defgikz^3 + 4eghkz^4, bcdgijkz + ceghjkz^2 + 4cefghz^3)\\
&\subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,k,z].\\
P_1 =&(x) \subset \mathbb{Q}[x,y,z].\\
P_2 =&(x_{13}, x_{23}, x_{33}, x_{43}) \subset \mathbb{Q}[x_{ij} \mid 1 \le i \le 4, 1 \le j \le 5].\\
P_3 =&(b,z) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,z].\\
P_4 =&(e,i,z) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,z].\\
P_5 =&(g,h,z) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,z].\\
P_6 =&(h,z) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,k,l,m,z].\\
P_7 =&(b,j,z) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,z].\\
P_8 =&(f,g,i) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,z].\\
P_9 =&(z^4 + hdb, c, g, k, l) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,k,l,z].\\
P_{10} =&(b,c,e,h,i,j) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,k,l,z].\\
P_{11} =&(e,k) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,k,l,z].\\
P_{12} =&(e,g,z) \subset \mathbb{Q}[b,c,d,e,f,g,h,z].\\
P_{13} =&(e,g,k,z) \subset \mathbb{Q}[b,c,d,e,f,g,h,i,j,k,z].\\
P_{14} =&(x,y) \subset \mathbb{Q}[x,y,z].\\
P_{15} =&(x_{12}x_{31} - x_{32}x_{11}, x_{42}x_{11} - x_{41}x_{12}, x_{42}x_{31} - x_{41}x_{32}, x_{44}x_{31} - x_{41}x_{34},\\
&x_{44}x_{32} - x_{42}x_{34}, x_{13}, x_{21}, x_{22}, x_{23}, x_{24}, x_{33}, x_{43})\\
&\subset \mathbb{Q}[x_{ij} \mid 1 \le i \le 4, 1 \le j \le 4].\\
P_{16} =&(x_{16}x_{27} - x_{17}x_{26}, x_{34}x_{13} - x_{33}x_{14}, x_{37}x_{16} - x_{36}x_{17}, x_{36}x_{27} - x_{37}x_{26},
\end{aligned}
$$

$$
\begin{aligned}
&x_{12}, x_{15}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{32}, x_{35}) \subset \mathbb{Q}[x_{ij} \mid 1 \le i \le 3, 1 \le j \le 7].\\
P_{17} =& (e, f, j, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, z].\\
P_{18} =& (c, d, j, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, z].\\
P_{19} =& (-4fec + 3d, b, g, h, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, z].\\
P_{20} =& (lfdb + 4higc, e, j, m) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, k, l, m, z].\\
P_{21} =& (c, d, h, j, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, z].\\
P_{22} =& (c, d, g, i, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, z].\\
P_{23} =& (b, c, d, e, f, g, h, i, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, k, l, z].\\
P_{24} =& (g, i, j, l, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, k, l, z].\\
P_{25} =& (f, g, k, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, k, l, z].\\
P_{26} =& (c, e, g, h, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, z].\\
P_{27} =& (c + 4jf, b, d, g, h, k, z) \subset \mathbb{Q}[b, c, d, e, f, g, h, i, j, k, z].
\end{aligned}
$$