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On an Implementation of Standard Bases, Gröbner Bases and Normal-form Using Algebraic Local Cohomology

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Abstract

In order to analyze properties of singularities, we sometimes need to compute standard bases and Gröbner bases. Therefore, a good implementation for computing them, is always required to study singularities. In this paper, we describe an implementation of standard bases, Gröbner bases and normal-form by using algebraic local cohomology. Moreover, we give a new algorithm for computing algebraic local cohomology. We show that the use of algebraic local cohomology provides efficient methods for computing standard bases, Gröbner bases and normal-form. Especially, for studying singularity theory, this implementation works powerfully.

1 Introduction

We describe an implementation of standard bases, Gröbner bases and normal-form by using algebraic local cohomology. In [24], Tajima and Nakamura studied the Jacobi ideals of isolated hypersurface singularities by using duality and describe, in particular, an effective method for solving membership problems for Jacobi ideals in local rings. In this paper, we give, using the same framework as in [24], a new method for treating standard bases and Gröbner bases of zero-dimensional ideals.

When we study singularities, we sometimes need to compute standard bases and Gröbner bases of given zero-dimensional Jacobi ideals. In this paper, we consider a Jacobi ideal of a given polynomial where the polynomial has an isolated singularity at the origin. We treat only zero-dimensional Jacobi ideals. Zero-dimensional ideals in the formal power series and the associated vector space consisting of algebraic local cohomology classes, are considered in the context of Grothendieck local duality ([9, 10]). This paper shows that the use of algebraic local cohomology provides an efficient method for computing standard bases and Gröbner bases. Since algebraic local cohomology classes have a lot of good properties and information for computing standard bases, Gröbner bases, normal-form and analyzing properties of singularities, first we consider an algorithm for computing

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algebraic local cohomology classes. In the first part of this paper, an algorithmic strategy for computing Čech cohomology representations of the algebraic local cohomology classes are described. In [1, 20, 21], Abe and Tajima illustrated an algorithm for computing algebraic local cohomology classes and it's applications. In this paper, we improve the algorithm and give applications.

In the second part of this paper, we describe applications of algebraic local cohomology classes, which are to compute standard bases, Gröbner bases and normal-form.

The theory of standard bases for ideals in power series rings was introduced in 1964 by Hironaka [11] on the resolution of singularities. Since then, standard bases have been extensively utilized in various fields. There are now two classical and widely used method that compute standard bases of ideals, in local rings, generated by polynomials. One method is based on Mora's tangent cone algorithm and the other is based on Lazard's homogenization technique [6, 7, 12, 14]. For the zero-dimensional case, there is an another approach, called duality method to deal with ideals in local rings, which has also been extensively studied by several authors in the context of computer algebra[2, 13, 16]. In [20, 21], Tajima has adopted another classical duality, the Grothendieck duality on local residues, for treating standard bases of zero-dimensional ideals. The key ingredient in this approach is the concept of algebraic local cohomology. We show that the use of algebraic local cohomology provides an efficient method for computing standard bases. Moreover, as Tajima and Nakamura studied a method for solving membership problems for Jacobi ideals in local rings ([24]), it is easily possible to adapt this method to compute normal-forms in local rings.

The use of Gröbner basis computations for treating of polynomial equations, has become an important tool in many areas, reaching from pure mathematics to industrial applications. Despite the well-known complexity of Buchberger's algorithm ([4]), it has turned out to be very practicable for many special cases of interest. One of the applications of algebraic local cohomology classes, is to compute Gröbner bases. Let f be a polynomial in $K[x_1, \dots, x_n]$ where K is the field of complex numbers or rational number, and $x_1, \dots, x_n$ are variables. Assume that f has an isolated singularity at the origin. Then, the zero-dimensional Jacobi ideal J of f is generated by $\partial f/\partial x_1, \dots, \partial f/\partial x_n$ where ∂ is a partial derivative. Now, consider a primary ideal decomposition of J. Assume that $J = J_1 \cap \cdots \cap J_m$ is a primary ideal decomposition of J where $J_1, \dots, J_m$ are primary ideals in $K[x_1, \dots, x_n]$. By using algebraic local cohomology classes, we can easily compute a Gröbner bases of J_i such that J_i has an isolated singularity at the origin ($\exists i \in \{1, \dots, m\}$).

The key ingredient of all applications in this paper, is the concept of algebraic local cohomology. The methods have the following advantages.

- **The algorithm ends up only with linear algebra.**
- We do not need Mora's reduction (tangent cone algorithm) for computing standard bases.
- We do not need S-polynomials for computing Gröbner bases.

That's why we can obtain them efficiently.

All algorithms of this paper, have been implemented in the computer algebra system Risa/Asir ([19]) by the author.

The outline of this paper is as follows: Section 2 introduces a representation of algebraic local cohomology classes using Čech cohomology classes ([8, 18]). Section 3 gives fundamental tools to construct our algorithms. Section 4 describes to compute algebraic local cohomology classes. Section 5, 6, 7 treat applications of algebraic local cohomology, which are to compute standard bases, normal-form and Gröbner bases. Section 8 compares our implementation to Abe's ([1]) implementation.

2 Preliminary

We fix the following notations throughout this paper. $\mathbb{Q}$, $\mathbb{C}$ and $\mathbb{N}$ are defined as the field of rational numbers, the field of complex numbers and the set of natural numbers, respectively. Note that in this paper, the set of natural number $\mathbb{N}$ includes zero. **We use the notation x as the abbreviation of n variables $x_1, \ldots, x_n$. I.e., $x = (x_1, \ldots, x_n)$.**

Let X be a neighborhood of the origin O of the n-dimensional complex space $\mathbb{C}^n$ and let $\mathcal{O}_X$ be the sheaf on X of holomorphic differential n-forms. Let $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ be the algebraic local cohomology supported at the origin O. Then, the space $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ has a structure of Fréchet-Schwartz topological vector space ([3]). Recall that the topological vector space $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ and the space $\mathcal{O}_{X,O}$ of formal power series at the origin, are mutually strong dual via the Grothendieck local residue pairing. This implies in particular the following fact which plays an important role in our approach ([23]).

" Any algebraic local cohomology class in $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ can be regarded as an analytic linear functional that acts on the space of formal power series at the origin."

Let $f(x)$ be a holomorphic function on $\mathcal{O}_X$. Consider the ideal $\mathcal{I}_O$ generated by $\frac{\partial f}{\partial x_1}(x)$, $\frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_n}(x)$ on $\mathcal{O}_{X,O}$. Then, the following set $\mathcal{H}_f$ is annihilated by $\mathcal{I}_O$ and a subset of $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ (algebraic local cohomology):

$$\mathcal{H}_f = \left\{ \psi \in \mathcal{H}^n_{[O]}(\mathcal{O}_X) \,\middle|\, \frac{\partial f}{\partial x_1}(x)\psi = \frac{\partial f}{\partial x_2}(x)\psi = \cdots = \frac{\partial f}{\partial x_n}(x)\psi = 0 \right\}.$$

Algebraic local cohomology $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ can be represented as, by taking X sufficiently small if necessary, an element of relative Čech cohomology. Therefore, to regard $\left[\frac{1}{x^{\lambda+1}}\right]$ as a relative Čech cohomology class, any algebraic local cohomology class in $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ can be represented as a finite sum of the form $\sum_{\lambda} c_\lambda \left[\frac{1}{x^{\lambda+1}}\right]$ ($c_\lambda \in \mathbb{C}, \lambda = (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n$) where $\lambda + 1 = (l_1+1, l_2+1, \ldots, l_n+1), 1 = (1, 1, \ldots, 1)$). We also use the notation $\sum \left[\frac{1}{x^{\lambda+1}}\right]$ or $\left[\sum \frac{1}{x^{\lambda+1}}\right]$ for representing algebraic local cohomology classes in $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$. Note that the multiplication is defined as

$$x^\kappa \left[\frac{1}{x^{\lambda+1}}\right] = \begin{cases} \left[\frac{1}{x^{\lambda+1-\kappa}}\right] & l_i \geq k_i, i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

where $\kappa = (k_1, \ldots, k_n) \in \mathbb{N}^n, \lambda = (l_1, \ldots, l_n) \in \mathbb{N}^n$, and $\lambda + 1 - \kappa = (l_1 + 1 - k_1, \ldots, l_n + 1 - k_n)$ (see [17, 22]). In the following example, we see a concrete example of $\mathcal{H}_f$.

Example 1

Let $f = x^3 + xy^3$ be a polynomial in $\mathbb{C}[x, y]$. Then, by $\frac{\partial f}{\partial x} = 3x^2 + y^3$, $\frac{\partial f}{\partial y} = 3xy^2$, we can easily check that the following elements belong to $\mathcal{H}_f$:

$$\left[\frac{1}{xy}\right], \left[\frac{1}{xy^2}\right], \left[\frac{1}{x^2y}\right], \left[\frac{1}{xy^3}\right], \left[\frac{1}{x^2y^2}\right] \in \mathcal{H}_f.$$

Furthermore, the following two elements also belong to $\mathcal{H}_f$*:*

$$\left[\frac{1}{xy^4}\right]-\frac{1}{3}\left[\frac{1}{x^3y}\right],\left[\frac{1}{xy^5}\right]-\frac{1}{3}\left[\frac{1}{x^3y^2}\right]\in\mathcal{H}_f.$$

Let us assume that a set $\left\{x\in X \,\middle|\, \frac{\partial f}{\partial x_1}(x)=\cdots=\frac{\partial f}{\partial x_n}(x)=0\right\}=\{O\}$ is given and K is $\mathbb{Q}$ or $\mathbb{C}$. We introduce a vector space $H^n_{[O]}(K[x])$ to be the set of algebraic local cohomology classes $\sum_\lambda c_\lambda\left[\frac{1}{x^{\lambda+1}}\right]$ with coefficients c_λ in K,

$$H^n_{[O]}(K[x])=\lim_{k\to\infty}Ext^n_{K[x]}(K[x]/\langle x_1,x_2,\ldots,x_n\rangle^k,K[x]).$$

We define a vector space H_f to be the set of algebraic local cohomology classes in $H^n_{[O]}(K[x])$ that are annihilated by these n polynomials $\frac{\partial f}{\partial x_1}(x),\ldots,\frac{\partial f}{\partial x_n}(x)$,

$$H_f=\left\{\psi\in H^n_{[O]}(K[x]) \,\middle|\, \frac{\partial f}{\partial x_1}(x)\psi=\frac{\partial f}{\partial x_2}(x)\psi=\cdots=\frac{\partial f}{\partial x_n}(x)\psi=0\right\}.$$

Then, the vector space H_f is the dual vector space of $K[[x]]/\mathcal{I}_O$, where $K[[x]]$ stands for the space of formal power series with coefficients in K and $\mathcal{I}_O$ is the ideal, in $K[[x]]$, generated by $\frac{\partial f}{\partial x_1}(x),\ldots,\frac{\partial f}{\partial x_n}(x)$. Since the duality is induced by local residues, we have the non-degenerate residue pairing,

$$\mathrm{res}_O(\ ,\ ):K[[x]]/\mathcal{I}_O\times H_f\longrightarrow K,$$

which can be regard as a special case of the Grothendieck local duality ([9]).

The non-degeneracy of the pairing implies the fact that, a formal power series $h\in K[[x]]$ is in the ideal $\mathcal{I}_O$ if and only if $\mathrm{res}_O(h,\phi)=0,\ \forall\phi\in H_f$. Thus, the ideal $\mathcal{I}_O$ is completely determined by the space H_f of algebraic local cohomology classes via local residues ([24]).

In order to compute algebraic local cohomology efficiently, we represent algebraic local cohomology class $\sum c_\lambda\left[\frac{1}{x^{\lambda+1}}\right]$ as a n variables polynomial $\sum c_\lambda\xi^\lambda$ where $\xi=(\xi_1,\xi_2,\ldots,\xi_n)$. We call this representation "**polynomial representation**". For example, let $\psi=\left[\frac{4}{x^3y^4}\right]+\left[\frac{5}{x^2y^3}\right]$ where x,y are variables. Then, the polynomial representation of ψ is $4\xi^2\eta^4+5\xi\eta^2$ where variables (ξ,η) are corresponding to variables (x,y). That is, we have the following table for two variables:

Čech representation		polynomial representation
$c_{(l,m)}\left[\frac{1}{x^{l+1}y^{m+1}}\right]$	$\longleftrightarrow$	$c_{(l,m)}\xi^l\eta^m$
$\sum c_{(l,m)}\left[\frac{1}{x^{l+1}y^{m+1}}\right]$	$\longleftrightarrow$	$\sum c_{(l,m)}\xi^l\eta^m$

where $c_{(l,m)}\in K$.

The multiplication for polynomial representation, is defined as follows:

$$x^\kappa * \xi^\lambda = \begin{cases} \xi^{\lambda-\kappa} & l_i \geq k_i, i = 1, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\kappa = (k_1, \ldots, k_n) \in \mathbb{N}^n, \lambda = (l_1, \ldots, l_n) \in \mathbb{N}^n$, and $\lambda - \kappa = (l_1 - k_1, \ldots, l_n - k_n)$. If we apply polynomial representation to Example 1, then we have $1, \eta, \xi, \eta^2, \xi\eta, \eta^3 - \frac{1}{3}\xi^2, \eta^4 - \frac{1}{3}\xi^2\eta \in H_f$.

In section 4, we show an algorithm for computing a basis of H_f (algebraic local cohomology) which is the main result of this paper.

3 Fundamental tools

In this section, we give some fundamental tools to construct algorithms for computing a basis of H_f, standard bases and Gröbner bases. First, we define a term order for algebraic local cohomology. Second, we give some notations of this paper.

3.1 Term ordering on algebraic local cohomology

In this paper, we use the following term order to compute a basis of H_f.

Definition 2 (term order)
For two multi-indices $\lambda = (l_1, \ldots, l_n)$ and $\lambda' = (l'_1, \ldots, l'_n)$ in $\mathbb{N}^n$, we denote

$$\left[\frac{1}{x^{\lambda'+1}}\right] \prec \left[\frac{1}{x^{\lambda+1}}\right] \text{ or } \lambda' + 1 \prec \lambda + 1$$

if $|\lambda' + 1| < |\lambda + 1|$, or $|\lambda' + 1| = |\lambda + 1|$ and there exists $j \in \mathbb{N}$ so that $l'_i + 1 = l_i + 1$ for $i < j$ and $l'_j + 1 < l_j + 1$, where $|\lambda| = l_1 + \cdots + l_n$.

Let consider the case of "polynomial representation". If $\left[\frac{1}{x^{\lambda'+1}}\right] \prec \left[\frac{1}{x^{\lambda+1}}\right]$, then obviously $\xi^{\lambda'} \prec \xi^\lambda$ in $K[\xi]$. In polynomial representation, this is the **total-degree lexicographic order** *such that $\xi_1 \prec \xi_2 \prec \xi_3 \prec \cdots \prec \xi_n$.*

For a given algebraic local cohomology class φ of the form $(\lambda', \lambda \geq 1)$

$$\varphi = c_\lambda \left[\frac{1}{x^\lambda}\right] + \sum_{\lambda' \prec \lambda} c_{\lambda'} \left[\frac{1}{x^{\lambda'}}\right], \ c_\lambda \neq 0,$$

we call $\left[\frac{1}{x^\lambda}\right]$ the **head term** (written: $\mathrm{ht}(\varphi)$) and $\left[\frac{1}{x^{\lambda'}}\right]$ the **lower term**. In polynomial representation, we also call $\xi^{\lambda-1}$ the **head term** (written: $\mathrm{ht}(\varphi) = \xi^{\lambda-1}$) and $\xi^{\lambda'-1}$ the **lower term**.

3.2 Notations and Functions

In this paper, we treat lists and sets as saving data for computing algebraic local cohomology. In the next definition, we introduce functions of lists and sets.

Definition 3
Let A be a set. A function list *converts A into the list* list(A) *whose elements are from A. This function does not care the ordering of A's elements in the list. Conversely, let B be a list. A function* set *converts B intor the set* set(B) *whose elements are from B.*

The set of lists over A, LIST(A), *is defined as the smallest set containing the empty list* [ ] *(different from any element of A) and the list $[a_1, \ldots, a_m]$ for all $a_1, \ldots, a_m \in A \cup$* LIST$(A)$. list$(A)$ *is equipped with the following (partial) operations:*
CONS*: $A \cup \mathrm{LIST}(A) \times \mathrm{LIST}(A) \rightarrow \mathrm{LIST}(A)$ maps $(a, [a_1, \ldots, a_n])$ to $[a, a_1, \ldots, a_n]$.*
CAR*: $\mathrm{LIST}(A) \rightarrow \mathrm{LIST}(A) \cap A$ maps $[a_1, \ldots, a_n]$ to a_1 and is undefined for* [ ].
CDR*: $\mathrm{LIST}(A) \rightarrow \mathrm{LIST}(A)$ maps $[a_1, \ldots, a_n]$ to $[a_2, \ldots, a_n]$ and is undefined for* [ ].
APPEND*: $\mathrm{LIST}(A) \times \mathrm{LIST}(A) \rightarrow \mathrm{LIST}(A)$ maps $([a_1, \ldots, a_n], [b_1, \ldots, b_m])$ to $[a_1, \ldots, a_n, b_1, \ldots, b_m]$.*

Let $A = [1, 3, 5, 7, 9]$ be a list. Then, $\mathrm{set}(A) = \{1, 3, 5, 7, 9\}$, $\mathrm{CAR}(A) = 1$, $\mathrm{CDR}(A) = [3, 5, 7, 9]$, $\mathrm{CONS}(10, A) = [10, 1, 3, 5, 7, 9]$ and $\mathrm{APPEND}([2, 4], A) = [2, 4, 1, 3, 5, 7, 9]$. Let $B = \{2, 4, 6, 8\}$ be a set. Then, $\mathrm{list}(B) = [2, 4, 6, 8]$.

A power product (or term) in $x_1, \ldots, x_n$ is an expression of the form $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for some exponent vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Then, the **degree** of this power product (written: $\deg(x^\alpha)$) is $(\alpha_1, \ldots, \alpha_n)$. The **total degree** of this power product (written : $\mathrm{tdeg}(x^\alpha)$ or $\mathrm{tdeg}(\alpha)$) is the sum $\alpha_1 + \cdots + \alpha_n$. Let $e_1, \ldots, e_r$ be the canonical basis of a free module. I.e, for each $i = 1, \ldots, r$,

$$e_i = \big(0, \ldots, 0, \overset{i\text{-th}}{1}, 0, \ldots, 0\big)$$

denotes the i-th canonical basis vector of $\mathbb{N}^r$ with 1 at the i-th place.

4 Computation Method (Algebraic Local Cohomology)

In this section, we introduce an algorithm for computing algebraic local cohomology classes. This section is the main part of this paper [25]. As we described in section 1, algebraic local cohomology has a lot of good properties and information for computing standard bases and analyzing properties of singularities. Hence, first, we need an algorithm for computing algebraic local cohomology.

This section is organized as follows. In subsection 4.1, we give the outline of an algorithm for computing algebraic local cohomology. In subsection 4.2, we describe the detail of the first step of the algorithm. The main results are shown in subsection 4.3 and 4.4. In subsection 4.3, we mention how to decide a head term of an element of a basis of H_f. In subsection 4.4, we describe how to decide lower terms of an element of a basis of H_f.

4.1 Outline

Before describing the detail, we give the outline of the algorithm for computing algebraic local cohomology (a basis of the vector space H_f). This outline facilitates understanding the detail of the whole algorithm. In the next subsection, we illustrate the detail.

Let f be a polynomial in $K[x]$. The algorithm for computing a basis of H_f, has mainly two steps. First, we need to compute elements of a basis of H_f which form monomial. After this step, a natural strategy for computing a basis of H_f is thus to find algebraic local cohomology classes from bottom to top by executing the following **STEP 2**.

OUTLINE

Input: f: a polynomial in $K[x]$,
Output: a basis of H_f (algebraic local cohomology classes).

STEP 1: To compute monomial elements.

Compute all monomial elements $\left[\frac{1}{x^{\lambda+1}}\right]$ of H_f where $\lambda \in \mathbb{N}^n$.

I.e., $\left(\frac{\partial f}{\partial x_1}\right)\left[\frac{1}{x^{\lambda+1}}\right] = \left(\frac{\partial f}{\partial x_2}\right)\left[\frac{1}{x^{\lambda+1}}\right] = \cdots = \left(\frac{\partial f}{\partial x_n}\right)\left[\frac{1}{x^{\lambda+1}}\right] = 0.$

STEP 2: To compute elements which form linear combination $\left(\sum c_\lambda \left[\frac{1}{x^{\lambda+1}}\right]\right)$.

1. Find a candidate λ of a head term.
2. Construct an associate set L_λ of candidates of lower terms
3. Set $p = \left[\frac{1}{x^{\lambda+1}}\right] + \sum_{\lambda' \in L_\lambda} c_{(\lambda,\lambda')} \left[\frac{1}{x^{\lambda'+1}}\right]$ $(\lambda, \lambda' \geq 0)$.
4. Check the condition $\left(\frac{\partial f}{\partial x_1}\right) p = \left(\frac{\partial f}{\partial x_2}\right) p = \cdots = \left(\frac{\partial f}{\partial x_n}\right) p = 0$, and decide $c_{(\lambda,\lambda')}$. That is, solve the system of linear equations which are created by the condition.

   Repeat from 1 to 4 until we obtain the condition of the termination.

In the rest of this section, we describe the detail of the each part of OUTLINE. **Note that, from here, we treat "polynomial representation" for computing a basis of H_f in algorithms.** If we obtain a basis of H_f in polynomial representation, then we can easily transform them into their Čech representation by the definition.

4.2 STEP 1

First, we see **STEP 1** for obtaining monomial elements of a basis of H_f. By the definition of H_f and the multiplication, we can easily compute monomial elements of a basis of H_f as follows.

(Algorithm) STEP1(f)

Input: f: a polynomial in $K[x]$ ($\prec$: the total degree term order,)
Output: MList: monomial elements of a basis of H_f,
FList: a list of exponents, GList: a list of lists, ML: a list.

(1). Compute $\frac{\partial f}{\partial x_i} = \sum_\alpha a_{i,\alpha} x^\alpha$ $(i = 1, \ldots, n)$ and $A = \{x^\alpha \in K[x] | \ \exists i, \text{ s.t. } a_{i,\alpha} \neq 0\}$.

(2). Compute the reduced Gröbner basis G of $\langle A \rangle$ in $K[x]$ w.r.t. $\prec$.

(3). Select the lowest term g in G. FList $\leftarrow$ list($\{\deg(g)\}$)

(4). Set an ordered list GList $= [G_{i_1}, G_{i_2}, \ldots, G_{i_s}]$ where $G_{i_j} = [\deg(r_{j,1}), \deg(r_{j,2}), \ldots, \deg(r_{j,l_j})]$, $r_{j,k} \in G \backslash \{g\}$, $\mathrm{tdeg}(r_{j,k}) = i_j$, $i_1 < i_2 < \cdots < i_s$, $r_{j,1} \prec r_{j,2} \prec \cdots \prec r_{j,l_j}$ for each $1 \leq j \leq s$ and $1 \leq k \leq l_j$.

(5). MList $\leftarrow$Compute monomial elements which does not belong to $\langle G \rangle$.
ML $\leftarrow \{\deg(g) | g \in \text{MList}\}$

Clearly, for all $\eta \in$ MList, $\left(\frac{\partial f}{\partial x_1}\right)\eta = \left(\frac{\partial f}{\partial x_2}\right)\eta = \cdots = \left(\frac{\partial f}{\partial x_n}\right)\eta = 0$. Therefore, by the definition of H_f, we can regard MList as a subset of a basis of H_f. In the following example, we see how **STEP1** works.

Example 4
Let $f = x^3y + xy^4 + x^2y^3$ be a polynomial in $K[x, y]$, and $\prec$ be a term order (see Definition 2) such that $y \prec x$.

(1). We have $\frac{\partial f}{\partial x} = 3x^2y + y^4 + 2xy^3$, $\frac{\partial f}{\partial y} = x^3 + 4xy^3 + 3x^2y^2$ *and* $A = \{x^2y, y^4, xy^3, x^3, xy^3, x^2y^2\}$.

(2). $G = \{x^2y, x^3, y^4, xy^3\}$ *is the reduced Gröbner basis of* $\langle A \rangle$ *w.r.t.* $\prec$.

(3). In G, x^2y is the lowest element w.r.t. $\prec$. Hence, FList = [(2, 1)].

(4). We have only one element x^3 whose total degree is 3. Elements of total degree 4 are y^4 and xy^3. Hence, GList = [[(3, 0)], [(0, 4), (1, 3)]].

(5). Finally, we need to commute a basis MList *of* $K[x, y]/\langle G \rangle$. *Then,* MList $= [1, \xi, \xi^2, \eta, \xi\eta, \eta^2, \xi\eta^2, \eta^3]$. *In Figure 1, a symbol "•" means an exponent of an element of* ML. *A symbol "∗" means an exponent of an element of* FList.

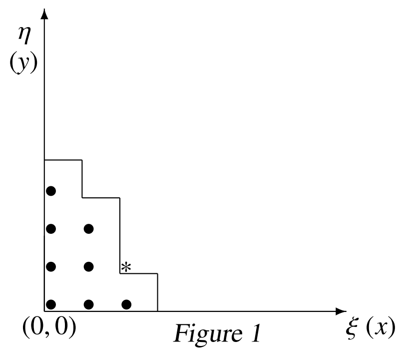

Figure 1

4.3 How to decide head terms.

In this and the next subsections, we consider **STEP 2** of **OUTLINE**. In **STEP 2**, we have to compute elements of a basis of a vector space H_f which form $\sum c_\lambda \left[\frac{1}{x^{\lambda+1}}\right]$. First, in this subsection, we describe how to decide the head terms. Second, in the next subsection, we describe how to decide the lower terms.

Let Λ_H denote the set of exponent of head terms in H_f and $\Lambda_H^{(\lambda)}$ denote a subset of Λ_H:

$$\Lambda_H = \{\lambda \in \mathbb{N}^n | \ \exists \varphi \text{ s.t. } \mathrm{ht}(\varphi) = \xi^\lambda, \varphi \in H_f\} \quad \text{and} \quad \Lambda_H^{(\lambda)} = \{\lambda' \in \Lambda_H | \lambda' \prec \lambda\}.$$

In order to construct the algorithms for computing $\sum c_\lambda \left[\frac{1}{x^{\lambda+1}}\right]$, let us recall the following results, which will be exploited several times in algorithms to compute a basis of H_f.

Lemma 5 ([20, 24])
If $\eta \in H_f$, so are $x_j\eta$, $j = 1, 2, \ldots, n$.

The Lemma 5 yields the followings [20, 24].

Lemma 6
Let $\lambda = (l_1, \ldots, l_n) \in \mathbb{N}^n$. If $\lambda \in \Lambda_H$, then, for each $j = 1, 2, \ldots, n$, $\lambda - e_j = (l_1, l_2, \ldots, l_{j-1}, l_j - 1, l_{j+1}, \ldots, l_n)$ is in $\Lambda_H^{(\lambda)}$.

Lemma 7
Let $\lambda = (l_1, \ldots, l_n) \in \mathbb{N}^n$. If $\lambda \notin \Lambda_H$, then, $(\lambda + \mathbb{N}^n) \cap \Lambda_H = \emptyset$. (I.e., $\exists \alpha \in \mathrm{FList}$, $x^\alpha * \xi^\lambda \neq 0$.)

Now, we construct an algorithm for computing a basis of H_f. In this subsection, we mainly mention how to decide head terms of a basis of H_f. In algorithms of this paper, we treat the following notations for saving data:

MList: a list of monomial elements of a basis of H_f (from **STEP 1**) in $K[\xi]$.
ML := $\{\deg(g) | g \in \mathrm{MList}\}$
SList: a list of non-monomial elements of a basis of H_f in $K[\xi]$.
TList := list($\{\deg(\mathrm{ht}(\varphi)) \in \mathbb{N}^n | \varphi \in \mathrm{SList}\}$) in $\mathbb{N}^n$.
FList: a list of ex-candidates of head terms in $\mathbb{N}^n$.
LList: a list (of exponents) of all lower terms of SList in $\mathbb{N}^n$.
CT: a list of candidates of head terms in $\mathbb{N}^n$. ($\forall \alpha, \beta \in \mathrm{CT}$, $\mathrm{tdeg}(\alpha) = \mathrm{tdeg}(\beta)$)
TT := list($\{\alpha | \alpha \in \mathrm{CT}, \alpha \in \mathrm{TList}\}$). ($\forall \alpha, \beta \in \mathrm{TT}$, $\mathrm{tdeg}(\alpha) = \mathrm{tdeg}(\beta)$)
CL: a list of candidates of lower terms in $\mathbb{N}^n$.

Therefore, the list APPEND(MList, SList) means a basis of H_f. When we create a list of lower terms LList, we need lists LL, UU, RR and EL in the following algorithm.

In the next subsection, we explain these lists and how to create the list CL.

(Algorithm) MAIN(GList, FList, ML)

Input: GList, FList, ML: lists (from **STEP 1**)
Output: SList, LList, TList, FList
(0). TT ← [ ]; CT ← CAR(GList); GList ← CDR(GList); TList ← [ ]
LList ← FList; EL ← FList; LL ← [ ]; RR ← [ ]; UU ← [ ] ($\diamond$1)
(1). if CT ≠ [ ] **then**
γ ← CAR(CT), CT ← CDR(CT) /* Select the lowest exponent γ in CT. */
else if CT = [ ] **then**
(CT, GList) ← `Head_candidate`(TT, CT, GList) ($*$1)
/* candidates of head terms.*/
if CT = [ ] **then** ($*$2) /* The condition of the termination */
return(SList, TList, LList, FList)

end-if
TT ← [ ]; γ ← CAR(CT) /* Select the lowest exponent γ in CT. */
CT ← CDR(CT)
end-if
(2) We consider that ξ^γ is a candidate of a head term.
/*Make a list CL of candidates of the lower terms. */
(CL, UU, EL) ← LOW_candidate(γ, LList, LL, UU, RR, EL, ML, TList) (◇2)
(3) Set $p = \xi^\gamma + \sum_{\lambda \in \mathrm{CL}} c_\lambda \xi^\lambda$ a candidate of a basis of H_f where $c_\lambda \in K$. Create a system of c_λ's linear equations from the condition $\left(\frac{\partial f}{\partial x_1}\right) * p = \left(\frac{\partial f}{\partial x_2}\right) * p = \cdots = \left(\frac{\partial f}{\partial x_n}\right) * p = 0$. Then, solve the system.

if a solutions of $c_\lambda s$ exists **then**
$Z \leftarrow 1$
TList ← CONS(γ, TList)
TT ← CONS(γ, TT)
p' ← substitute the solution into $c_\lambda s$ of p.
SList ← CONS(p', SList) /* p' is an element of the basis. */
else
$Z \leftarrow 0$; $p' \leftarrow 0$
FList ← CONS(γ, FList) /*ξ^γ is not a head term of the basis. */
end-if
(LList, LL, RR, EL) ← renew(Z, γ, p', LList, EL) (◇3)
Repeat from (1) to (3) until (∗2) **comes.**

Remark: In the next subsection, we explain (◇1), (◇2) and (◇3) for making candidates of lower terms. I.e., we see the algorithms LOW_candidate and renew. In this subsection, we do not describe them.

In (1), if CT = [ ], then we need to renew the list CT (see (∗1)). In order to renew the list, we have to consider the following four cases.

Case (i) TT = [ ], GList = [ ]. **Case (ii)** TT = [ ], GList ≠ [ ].
Case (iii) TT ≠ [ ], GList = [ ]. **Case (iv)** TT ≠ [ ], GList ≠ [ ].

The next algorithm H_candidate tells us how to renew the list CT in each case. Actually, if we have the condition "TT = [ ] and GList = [ ]", the algorithm MAIN terminates. In the next algorithm, we apply Lemma 5,6,7 to make a list of candidates of head terms. The following functions are required in the algorithm H_candidate, actually which are from Lemma 5,6,7.

(Function) nb(TT)

Input TT, **Output** S: a list.
$S \leftarrow [\ ]$
while TT $\neq [\ ]$ **then**
$\tau \leftarrow$ CAR(TT); TT $\leftarrow$ CDR(TT)
  for i from 1 **to** n **do** $\alpha \leftarrow \tau + e_i$
    if $\alpha \notin S$ **then** $S \leftarrow$ CONS(α, S) **end-if**
  end-for
end-while
return(S)

(Function) cf(L, FList)

Input L: a list of elements in $\mathbb{N}^n$, FList, **Output** S: a list.
$S \leftarrow [\ ]$
while $L \neq [\ ]$ **do** $t \leftarrow 1$
$\alpha \leftarrow$ CAR(L); $L \leftarrow$ CDR(L); $W \leftarrow$ FList
  while $W \neq [\ ]$ **do**
    $\kappa \leftarrow$ CAR(W); $W \leftarrow$ CDR(W)
    if $x^\kappa * \xi^\alpha \neq 0$ **then** $t \leftarrow 0$; break **end-if**
  end-while
  if $t = 1$ **then** $S \leftarrow$ CONS(α, S) **end-if**
end-while
return(S)

We give an example of the functions. Let $L = [(1,2),(3,1),(3,4),(1,8)]$ and FList $= [(2,3),(4,0)]$. Then, nb(L) $= [(2,2), (1,3),(4,1), (3,2),(4,4), (3,5),(2,8), (1,9)]$ and cf(L, FList) $= [(1,2), (3,1), (1,8)]$.

Now, we are ready to construct the following algorithm which creates candidates of head terms.

(Algorithm) H_candidate(TT, CT, GList)

Input: TT, CT, GList: lists from MAIN
Output: CT, GList: lists
Case (i) TT = [ ], GList = [ ]
    return([ ], [ ]) /* termination */
Case (ii) TT = [ ], GList ≠ [ ]
    CT ← CAR(GList); GList = CDR(GList); return(CT, GList)
Case (iii) TT ≠ [ ], GList = [ ]
    NT ← nb(TT); CT ← cf(NT, FList); return(CT, GList)
Case (iv) TT ≠ [ ], GList ≠ [ ]
Set $dt \in \mathbb{N}$ as the total degree of TT. (All elements of TT have the same total degree.) Set $dg \in \mathbb{N}$ as the total degree of CAR(GList). (All elements of CAR(GList) have the same total degree, too.) In general, $dg > dt$ ($dg \neq dt$).
if $dg - dt > 1$ **then**
    NT ← nb(TT); CT ← cf(NT, FList); return(CT, GList)
end-if
if $dg - dt = 1$ **then**
    CT ← cf(nb(TT), FList) ∪ CAR(GList); GList ← CDR(GList)
    return (CT, GList)
end-if
Assume that after creating lists, all elements of the lists are lined up in order of $\prec$.

When the list CT can not be renewed, the algorithm MAIN terminates. That is, **case (i)** "TT = [ ], GList = [ ]" is the condition of the termination. The following example tells us how the algorithms work.

Example 8
Let consider Example 4, again. Let $f = x^3y + xy^4 + x^2y^3$ be a polynomial in $K[x, y]$. In Example 4, we obtained FList = [(2, 1)], GList = [[(3, 0)], [(0, 4), (1, 3)]]. *We apply the algorithm* MAIN *for computing a basis of H_f. We do not describe (◇1), (◇2), and (◇3). In Example 10, we see the computation of making candidates of lower terms.*

1-0. TT = [ ]; CT = CAR(GList) = [(3, 0)]; GList = CDR(GList) = [[(0, 4), (1, 3)]].

1-1. *By* MAIN (1), γ = CAR(CT) = (3, 0) *and* CT = CDR(CT) = [ ]. *Now, ξ^3 is a candidate of a head term. In* (2), LOW_candidate *outputs a list* CL *as a set of candidates of the lower terms. In the next subsection, we describe how compute* CL. *In* (3), *set $p = \xi^3 + \sum_{\lambda \in \mathrm{CL}} c_\lambda \xi^\lambda$ and solve the system of linear equations which is from $\left(\frac{\partial f}{\partial x}\right) * p = \left(\frac{\partial f}{\partial y}\right) * p = 0$. Then, there does not exist a solution of c_λ. Hence, a basis of H_f does not have an element whose head term is ξ^3.* FList = CONS((3, 0), FList) = [(3, 0), (2, 1)].

2-0. *We have* CT = [ ], TT = [ ] *and* GList = [[(0, 4), (1, 3)]]. *By* H_candidate (ii), CT = [(0, 4), (1, 3)], GList = CDR(GList) = [ ].

2-1. *In* (1), TT = [ ], γ = CAR(CT) = (0, 4) *and* CT = [(1, 3)]. *η^4 is a candidate of a head term.* LOW_candidate *outputs* CL. *Set $p = \eta^4 + \sum_{\lambda \in \mathrm{CL}} c_\lambda \xi^\lambda$ and solve the system of linear equations which is from $\left(\frac{\partial f}{\partial x}\right) * p = \left(\frac{\partial f}{\partial y}\right) p = 0$. Then, there exists a solution. Therefore, we have $p_1 = \eta^4 - \frac{1}{3}\xi^2\eta$ as an element of a basis of H_f and* TT = CONS((0, 4), TT) = [(0, 4)], SList = CONS(p_1, SList).

3-1. *As* $\mathrm{CT} = [(1,3)]$ *in* (1), $\gamma = \mathrm{CDR(CT)} = (1,3)$ *and* $\mathrm{CT} = \mathrm{CDR(CT)} = [\ ]$. $\xi\eta^3$ *is a candidate of a head term.* LOW_candidate *outputs* CL. *Set* $p = \xi\eta^3 + \sum_{\lambda\in\mathrm{CL}} c_\lambda \xi^\lambda$ *and solve the system of linear equations. Then, there exists a solution. Therefore, we have* $p_2 = \xi\eta^3 - 4\xi^3 - \frac{2}{3}\xi^2\eta$ *and* $\mathrm{TT} = \mathrm{CONS}((1,3),\mathrm{TT}) = [(1,3),(0,4)]$, $\mathrm{SList} = \mathrm{CONS}(p_2,\mathrm{SList})$.

4-0. *As* $\mathrm{CT} = [\ ]$, $\mathrm{TT} = [(1,3),(0,4)] \neq [\ ]$ *and* $\mathrm{GList} = [\ ]$, *by* H_candidate (iii), *we have* $\mathrm{NT} = \mathrm{nb((TT))} = [(0,5),(1,4),(2,3)]$ *and* $\mathrm{CT} = \mathrm{cf(NT,FList)} = [(0,5),(1,4)]$.

4-1. *In* (1), $\mathrm{TT} = [\ ]$, $\mathrm{CT} = [(0,5),(1,4)]$, $\gamma = \mathrm{CAR(CT)} = (0,5)$ *and* $\mathrm{CT} = \mathrm{CDR(CT)} = [(1,4)]$. η^5 *is a candidate of a head term.* LOW_candidate *outputs* CL. *Set* $p = \eta^5 + \sum_{\lambda\in\mathrm{CL}} c_\lambda \xi^\lambda$ *and solve the system of linear equations. Then, there exists a solution. Therefore, we obtain* $p_3 = \eta^5 - \frac{1}{3}\xi^2\eta^2 + \xi^3$ *and* $\mathrm{TT} = \mathrm{CONS}((0,5),\mathrm{TT}) = [(0,5)]$, $\mathrm{SList} = \mathrm{CONS}(p_3,\mathrm{SList})$.

5-1. *As* $\mathrm{CT} = [(1,4)]$ *in* (1), $\gamma = \mathrm{CAR(CT)} = (1,4)$ *and* $\mathrm{CT} = \mathrm{CDR(CT)} = [\ ]$. $\xi\eta^4$ *is a candidate of a head term.* LOW_candidate *outputs* CL. *Set* $p = \xi\eta^4 + \sum_{\lambda\in\mathrm{CL}} c_\lambda \xi^\lambda$ *and solve the system of linear equations. Then, there does not exist a solution.* $\mathrm{FList} = \mathrm{CONS}((1,4),\mathrm{FList}) = [(1,4),(3,0),(2,1)]$.

6-0. *We have* $\mathrm{CT} = [\ ]$, $\mathrm{TT} = [(0,5)]$ *and* $\mathrm{GList} = [\ ]$. *By* H_candidate (iii), $\mathrm{NT} = \mathrm{nb(TT)} = [(0,6),(1,5)]$ *and* $\mathrm{CT} = \mathrm{cf(NT,FList)} = [(0,6)]$.

6-1. *In (1),* $\mathrm{TT} = [\ ]$, $\gamma = \mathrm{CAR(CT)} = (0,6)$, $\mathrm{CT} = [(0,6)]$ *and* $\mathrm{CT} = \mathrm{CDR(CT)} = [\ ]$. η^6 *is a candidate of a head term.* LOW_candidate *outputs* CL. *Set* $p = \eta^6 + \sum_{\lambda\in\mathrm{CL}} c_\lambda \xi^\lambda$ *and solve the system of linear equations. Then, there exists a solution. Therefore, we obtain* $p_4 = \eta^6 - \frac{1}{3}\xi^2\eta^3 + \frac{7}{33}\xi\eta^4 + \frac{4}{3}\xi^4 + \frac{5}{33}\xi^2\eta - \frac{14}{99}\xi^2\eta^2 + \frac{14}{33}\xi^3$ *and* $\mathrm{TT} = \mathrm{CONS}((0,6),\mathrm{TT}) = [(0,6)]$, $\mathrm{SList} = \mathrm{CONS}(p_4,\mathrm{SList})$. *In Figure 8-1, a symbol "•" means an element of* ML *from Example 4. A symbol "∗" means of an element of* FList. p_i *means an exponent of* $\mathrm{ht}(p_i)$ *for each* $i = 1,2,3,4$.

7-0. *We have* $\mathrm{CT} = [\ ]$, $\mathrm{TT} = [(0,6)]$ *and* $\mathrm{GList} = [\ ]$. *By* H_candidate (iii), $\mathrm{NT} = \mathrm{nb(TT)} = [(0,7),(1,6)]$ *and* $\mathrm{CT} = \mathrm{cf(NT,FList)} = [(0,7)]$.

7-1. *In* (1), $\mathrm{TT} = [\ ]$, $\gamma = \mathrm{CAR(CT)} = (0,7)$ *and* $\mathrm{CT} = \mathrm{CDR(CT)} = [\ ]$. η^7 *is a candidate of a head term.* LOW_candidate *outputs* CL. *Set* $p = \eta^7 + \sum_{\lambda\in\mathrm{CL}} c_\lambda \xi^\lambda$ *and solve the system of linear equations. Then, does not exist a solution.* $\mathrm{FList} = \mathrm{CONS}((0,7),\mathrm{FList}) = [(0,7),(1,4),(3,0),(2,1)]$.

8-0. *Now,* $\mathrm{CT} = [\ ]$, $\mathrm{TT} = [\ ]$ *and* $\mathrm{GList} = [\ ]$. *By* H_candidate (i), CT *can not be renewed. This is the condition of the termination. The computation stops. Therefore, we obtain a basis of* H_f *which is* APPEND(SList, MList). *In Figure 8-2, a symbol "∗" means an element of* FList, *a symbol "△" means an element of* TT.

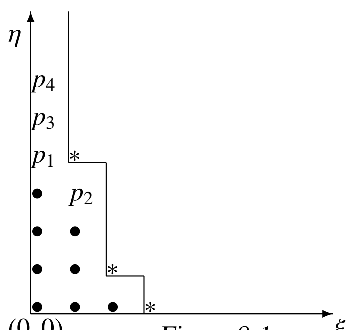

Figure 8-1

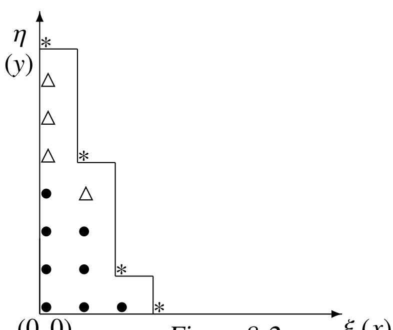

Figure 8-2

4.4 How to decide lower terms.

In this subsection, we describe how to decide lower terms. That is, we introduce the algorithms renew and LOW_candidate which are in the algorithm MAIN. In fact, we improve the algorithm [1, 20, 21] for computing a basis of H_f. A big improvement is a method of creating candidates of lower terms. While the algorithm runs for computing them, we can obtain a lot of good information ML, FList, TList, LList and SList. We are able to apply these information for getting a small number of the candidates. In [1], Abe has implemented an algorithm for computing a basis of H_f. Hoverer, he has not applied these information for computing them. This point is the big difference. In section 8, we compare Abe's implementation to our implementation. Before describing the algorithms, we introduce the following lemma which says properties of lower terms.

Lemma 9
Let p be an element of H_f and $\left[\frac{1}{x_1^{l_1+1}\cdots x_n^{l_n+1}}\right]$ be a lower term of p. During the computation of a basis of H_f (in the algorithm MAIN), exponents of the following terms

$$\left[\frac{1}{x_1^{l_1}x_2^{l_2+1}\cdots x_n^{l_n+1}}\right], \left[\frac{1}{x_1^{l_1+1}x_2^{l_2}x_3^{l_3+1}\cdots x_n^{l_n+1}}\right], \ldots, \left[\frac{1}{x_1^{l_1+1}x_2^{l_2+1}\cdots x_{n-1}^{l_{n-1}+1}x_n^{l_n}}\right]$$

belong to one of the followings:
(1) LList, (2) TList, (3) ML, (4) [0] (*become zero.*).

If we adopt polynomial representation, we can write them as the following.
Let $\xi_1^{l_1}\cdots\xi_n^{l_n}$ be a lower term of p. Then, exponents of the terms

$$(l_1-1, l_2, \ldots, l_n), (l_1, l_2-1, \ldots, l_n), \ldots, (l_1, \ldots, l_{n-1}-1, l_n), (l_1, \ldots, l_{n-1}, l_n-1)$$

(i) belong to set(ML) ∪ set(TList) ∪ set(LList), or

(ii) have Os (zeros) in some components.

We can write this lemma as the following function "cd".

(Function) cd(NL, ML, TList, LList)

Input: NL: a list of elements of $\mathbb{N}^n$, ML, TList, LList
Output: S: a list.
$S \leftarrow [\,]$
while NL $\neq [\,]$ **do**
  $\tau \leftarrow$ CAR(NL); NL $\leftarrow$ CDR(NL)
  if ($\tau \in$ set(ML) ∪ set(TList) ∪ set(LList)) or
    ($\exists i \in \mathbb{N}$ s.t. i-th component of τ is 0) **then**
   $S \leftarrow$ CONS(τ, S)
  end-if
end-while
return(S)

The algorithm LOW_candidate which is the main result of this subsetion, needs the functions "cd" and "nd", to create candidates of lower terms.

Now, let consider lower terms. As the vector space H_f is the dual vector space of $K[[x]]/J$, if lower terms belong to TList and ML, then the lower terms are reduced by MList and SList. Hence, we do not need the elements of ML and TList as lower terms. As we described, CL is a list of candidates of lower terms of γ. By this fact, CL can be written as CL = APPEND([new condiates of lower terms], LList). In algorithms of this paper, we treat the following notations for saving data:

EL: a list of new candidates of lower terms which does not belong to LList.

Let p is in SList. Then,

LL := list({ all lower terms of p} $\cap$ set(EL)).
RR := list(set(EL)$\setminus$ set(LL)).
UU := list({$\alpha \in$ nb(LL)|$\alpha > \gamma$}).
Note that CL = APPEND(EL, LList).

In MAIN (3), we have to renew the lists LList, UU and EL (see ($\diamond$ 3)). In order to renew the lists, we have to consider the two cases. One is the case $Z = 1$ (a solution exists). The other is the case $Z = 0$ (a solution does not exist). The next algorithm tells us how to renew the lists.

(Algorithm) renew(LList, EL, Z, γ, p')

Input: LList, EL: lists, Z: 0 or 1, γ: an element in $\mathbb{N}^n$, p': a polynomial in $K[\xi]$
(All arguments are from MAIN ($\diamond$3).)
Output: LList, LL, RR, EL : lists of elements of $\mathbb{N}^n$,
if $Z = 0$ **then**
LL ← [ ]; EL ← CONS(γ, EL); RR ← [ ]
else
LL ← list({All lower terms of p'} $\cap$ set(EL)) (♣1)
if LL ≠ [ ] **then**
LList ← APPEND(LL, LList); RR ← list(set(EL)$\setminus$ set(LL)); EL ← [ ]
end-if
end-if
return(LList, LL, RR, EL)

In order to create candidates of lower terms, we need to consider the following two cases.
Case (i) LL = [ ]. **Case (ii)** LL ≠ [ ].

In the algorithm renew, as LL = list({all lower terms of p'} $\cap$ set(EL)), LL has new lower terms which do not exist LList on (♣1). Therefore, if LL ≠ [ ], then, by Lemma 9, we have to create new candidates of lower terms for the next head term. In the case LL = [ ], we have to renew as the following algorithm.

(Algorithm) LOW_candidate(γ, LList, LL, UU, RR, EL, ML, TList)

Input: $\gamma \in \mathbb{N}^n$, LList, LL, UU, RR, EL, ML, TList : lists from MAIN,
Output: CL, UU, EL: lists, (Elements of CL are candidates of lower terms of γ.)
(Case 1) if LL = [ ] **then**
if UU ≠ [ ] **then**
E ← {α|$\gamma \succ \alpha$ (α is smaller than γ), $\alpha \in$ UU}
UU ← list(set(UU)$\setminus$ cd(E))
EL ← APPEND(list(cd(E)), EL)
end-if
end-if
CL ← APPEND(EL, LList)

```
  return(CL, UU, EL)
(Case 2) if LL ≠ [ ] then
      /* Make lists EL and UU by RR and LL */
        E ← {α|γ ≻ α, (α is smaller than γ), α ∈ UU}
        UU ← list((set(UU)\E)\{γ})    /*if γ ∈ UU */
        RR ← APPEND(list(E), RR)
        NL ← set(nb(LL))
        B ← {β| β ≻ γ (γ is smaller than β), β ∈ NL)
        UU ← APPEND(list(B), UU)
        D ← cd(list(NL \B), ML, TList, LList)
        EL ← APPEND(list(D\(D ∩ set(RR)), RR)
      end-if
    CL ← APPEND(EL, LList)
  return(CL, UU, EL)
```

We see an example how the algorithms renew and LOW_candidate work.

Example 10
Let consider Example 4, again. Let $f = x^3y + xy^4 + x^2y^3$ *be a polynomial in* $K[x, y]$*. In Example 8, we saw how to decide head terms. In this example, we see how to decide lower terms.*

0. *(Initialization) In* ($\diamond$1), LList = [ ], EL = [(2, 1)](= FList), UU = [ ], LL = [ ].

1. (3, 0) *is a candidate of a head term. As* LL = [ ], UU = [ ], *by* LOW_candidate **(Case 1)**, CL = APPEND(EL, LList) = [(2, 1)]. *Set* $p = \xi^3 + c_{(2,1)}\xi^2\eta$ *and check the conditon* $\left(\frac{\partial f}{\partial x}\right) * p = \left(\frac{\partial f}{\partial y}\right) * p = 0$. *That is, solve the systemof linear equation* $\left\{\left(\frac{\partial f}{\partial x}\right) * p = 3c_{(2,1)} = 0, \left(\frac{\partial f}{\partial y}\right) * p = 1 = 0\right\}$. *As there does not exist a solution of* $c_{(2,1)}$, *we obtain* $Z = 0$ *in* MAIN (3). *By* renew, *we have* LL = [ ], EL = CONS((3, 0), EL) = [(3, 0), (2, 1)] *and* LList = [ ].

2. (0, 4) *is a candidate of a head term. As* LL = [ ], UU = [ ], *by* LOW_candidate **(Case 1)**, CL = APPEND(EL, LList) = [(3, 0), (2, 1)]. *Set* $p = \eta^4 + c_{(3,0)}\xi^3 + c_{(2,1)}\xi^2\eta$ *and solve the system of linear equations* $\left\{\left(\frac{\partial f}{\partial x}\right) * p = 3c_{(2,1)} + 1 = 0, \left(\frac{\partial f}{\partial y}\right) * p = c_{(3,0)} = 0\right\}$. *Then, we have a solution* $c_{(3,0)} = 0$, $c_{(2,1)} = -\frac{1}{3}$. *Therefore,* $p = \eta^4 - \frac{1}{3}\xi^2\eta$. *By* renew *and* $Z = 1$, *we have* LL = [(2, 1)], LList = [(2, 1)], RR = [(3, 0)] *and* EL = [ ].

3. (1, 3) *is a candidate of a head term. As* LL = [(2, 1)], UU = [ ], *we apply* LOW_candidate **(Case 2)***:* E = { }, UU = [ ], RR = [(3, 0)], NL = {(3, 1), (2, 2)}, B = {(3, 1), (2, 2)} *(as* (1, 3) ≻ (3, 1), (2, 2)), UU = [(3, 1), (2, 2)] *(as* (1, 3) ≻ (3, 1), (2, 2)) *and* EL = [(3, 0)]. *Then,* CL = APPEND(EL, LList) = [(3, 0), (2, 1)]. *Set* $\xi\eta^3 + c_{(3,0)}\xi^3 + c_{(2,1)}\xi^2\eta$ *and solve the system of linear equations* $\left\{\left(\frac{\partial f}{\partial x}\right) * p = 3c_{(2,1)} + 2 = 0, \left(\frac{\partial f}{\partial y}\right) * p = c_{(3,0)} + 4 = 0\right\}$. *Then, we obtain a solution* $c_{(3,0)} = -4$, $c_{(2,1)} = -\frac{2}{3}$. *Therefore,* $p = \xi\eta^3 - 4\xi^3 - \frac{2}{3}\xi^2\eta$. *Since* (3, 0) ∈ EL *and* (2, 1) ∉ EL, *we obtain* LL = [(3, 0)], LList = [(3, 0), (2, 1)], RR = [ ] *and* EL = [ ].

4. (0, 5) *is a candidate of a head term. As* LL = [(3, 0)], UU = [(3, 1), (2, 2)], *we apply* LOW_candidate **(Case 2)***:* E = {(3, 1), (2, 2)}, UU = [ ], RR = [(3, 1), (2, 2)], NL = nb(LL) = {(4, 0), (3, 1)}, B = { } *(as* (0, 5) ≻ (4, 0), (3, 1)), UU = [ ], D = {(4, 0), (3, 1)} *and* EL = [(4, 0), (3, 1), (2, 2)]. *Hence,* CL = APPEND(EL, LList) = [(4, 0), (3, 1), (2, 2), (3, 0), (2, 1)]. *Set* $p = \eta^5 + c_{(4,0)}\xi^4 + c_{(3,1))}\xi^3\eta + c_{(2,2)}\xi^2\eta^2 + c_{(3,0)}\xi^3 + c_{(2,1)}\xi^2\eta$ *and check the condition* $\left(\frac{\partial f}{\partial x}\right) * p = 3c_{(3,1)}\xi + (3c_{(2,2)} + 1)\eta + 3c_{(2,1)} = 0$ *and* $\left(\frac{\partial f}{\partial y}\right) * p = c_{(4,0)}\xi + c_{(3,1)}\eta + (c_{(3,0)} + 3c_{(2,2)}) = 0$. *Then, we have a system of linear equations*

$\{3c_{(3,1)} = 0, 3c_{(2,2)} + 1 = 0, 3c_{(2,1)} = 0, c_{(4,0)} = 0, c_{(3,1)} = 0, c_{(3,0)} + 3c_{(2,2)} = 0\}$. *A solution of the system is* $c_{(4,0)} = 0$, $c_{(3,1)} = 0$, $c_{(2,2)} = -\frac{1}{3}$, $c_{(3,0)} = 1$, $c_{(2,1)} = 0$. *Therefore,* $p = \eta^5 - \frac{1}{3}\xi^2\eta^2 + \xi^3$. *Since* $(2,2) \in \mathrm{EL}$, *we obtain* $\mathrm{LL} = [(2,2)]$, $\mathrm{LList} = [(2,2),(3,0),(2,1)$, $\mathrm{RR} = [(4,0),(3,1)]$ *and* $\mathrm{EL} = [\ ]$.

5. $(1,4)$ *is a candidate of a head term. As* $\mathrm{LL} = [(2,2)], \mathrm{UU} = [\ ]$, *we apply* LOW_candidate **(Case 2)***:* $E = \{\ \}$, $\mathrm{RR} = [(4,0),(3,1)]$, $\mathrm{NL} = \mathrm{nb}(\mathrm{LL}) = [(3,2),(2,3)]$, $\mathrm{UU} = [(3,2),(2,3)]$ *(as* $(3,2),(2,3) \succ (1,4)$*),* $D = \{\ \}$ *and* $\mathrm{EL} = [(4,0),(3,1)]$. *Hence,* $\mathrm{CL} = [(4,0),(3,1),(2,2),(3,0),(2,1)]$. *Set* $p = \xi\eta^4 + c_{(4,0)}\xi^4 + c_{(3,1))}\xi^3\eta + c_{(2,2)}\xi^2\eta^2 + c_{(3,0)}\xi^3 + c_{(2,1)}\xi^2\eta$ *and check the condition* $\left\{\left(\frac{\partial f}{\partial x}\right) * p = 0, \left(\frac{\partial f}{\partial y}\right) * p = 0\right\}$. *Then, this system does not have a solution, and* $Z = 0$. *By* renew, $\mathrm{LL} = [\ ]$, $\mathrm{EL} = [(1,4),(4,0),(3,1)]$ *and* $\mathrm{RR} = [\ ]$.

6. $(0,6)$ *is a candidate of a head term. As* $\mathrm{LL} = [\ ], \mathrm{UU} = [(3,2),(2,3)]$, *we apply* LOW_candidate **(Case 1)***:* $E = \{(3,2),(2,3)\}$, $\mathrm{cd}(E) = \{(2,3)\}$, $\mathrm{UU} = [(3,4)]$, $\mathrm{EL} = [(2,3),(1,4),(4,0),(3,1)]$. *Hence,* $\mathrm{CL} = [(2,3),(1,4),(4,0),(3,1),(2,2),(3,0),(2,1)]$. *Set* $p = \eta^6 + c_{(2,3)}\xi^2\eta^3 + c_{(1,4))}\xi\eta^4 + c_{(4,0)}\xi^4 + c_{(3,1)}\xi^3\eta + c_{(2,2)}\xi^2\eta^2 + c_{(3,0)}\xi^3 + c_{(2,1)}\xi^2\eta$ *and check the condition* $\left\{\left(\frac{\partial f}{\partial x}\right) * p = 0, \left(\frac{\partial f}{\partial y}\right) * p = 0\right\}$. *Then, we have a solution* $c_{(2,3)} = -\frac{1}{3}, c_{(1,4)} = \frac{7}{33}, c_{(4,0)} = \frac{4}{3}, c_{(3,1)} = \frac{5}{33}, c_{(2,2)} = -\frac{14}{99}, c_{(3,0)} = \frac{12}{33}, c_{(2,1)} = 0$. *Therefore,* $p = \eta^6 - \frac{1}{3}\xi^2\eta^3 + \frac{7}{33}\xi\eta^4 + \frac{4}{3}\xi^4 + \frac{5}{33}\xi^2\eta - \frac{14}{99}\xi^2\eta^2 + \frac{14}{33}\xi^3$. *Since* $(2,3),(1,4),(4,0),(3,1)$ *in* EL, *we have* $\mathrm{LL} = [(2,3),(1,4),(4,0),(3,1)]$, $\mathrm{RR} = [\ ]$ *and* $\mathrm{LList} = [(2,3),(1,4),(4,0),(3,1),(2,2),(3,0),(2,1)]$.

7. $(0,7)$ *is a candidate of a head term. As* $\mathrm{LL} \neq [\ ]$ *and* $\mathrm{UU} = [(3,2)]$, *we apply* LOW_candidate **(Case 2)***:* $E = \{(3,2)\}$, $\mathrm{UU} = [\ ]$, $\mathrm{RR} = [(3,2)]$, $\mathrm{NL} = \mathrm{nb}(\mathrm{LL}) = [(3,3),(2,4),(1,5),(5,0),(4,1),(3,2)]$, $B = \{\ \}$, $D = \mathrm{NL}$, $\mathrm{EL} = [(3,3),(2,4),(1,5),(5,0),(4,1),(3,2)]$. *Hence,* $\mathrm{CL} = \mathrm{APPEND}(\mathrm{EL}, \mathrm{LList})$. *Set* $p = \eta^7 + \sum_{\lambda\in\mathrm{CL}} c_\lambda\xi^\lambda$ *and check the condition* $\left\{\left(\frac{\partial f}{\partial x}\right) * p = 0, \left(\frac{\partial f}{\partial y}\right) * p = 0\right\}$. *Then, does not exist a solution, and* $Z = 0$. *By* renew, $\mathrm{LL} = [\ ]$, $\mathrm{EL} = [(0,7),(3,3),(2,4),(1,5),(5,0),(4,1),(3,2)]$ *and* $\mathrm{RR} = [\ ]$.

Now, we can compute a basis of H_f (algebraic local cohomology). The following algorithm outputs lists MList, SList, TList, LList and FList. A basis of H_f is the list APPEND(MList, SList). As other lists are needed for computing standard bases and Gröbner bases, we let the next algorithm output them for the applications.

(Algorithm) ALC(f)

Input: f: a polynomial in $K[x]$,
Output: MList, SList, TList, LList, FList.
(APPEND(MList, SList) is a basis of H_f.)
(MList, ML, FList, GList) $\leftarrow$ **STEP1**(f)
(SList, TList, LList, FList) $\leftarrow$ MAIN(GList, FList, ML)
return(MList, SList, TList, LList, FList)

The algorithm ALC has been implemented on the computer algebra system Risa/Asir ([18]). This implementation outputs MList and SList as a basis of H_f.

Example 11
Let consider E_{12} *singularity defined by* $f = x^3 + y^7 + xy^5$. *Our implementation can output a polynomial representation of a basis of* H_f.

```
[467] cohomology(x^3+y^7+x*y^5,[x,y],1,0,0);
[[1,y,x,y^2,y*x,y^3,y^2*x,y^4,y^3*x],[-1/3*x^2+y^5,-1/3*y*x^2-7/5
*y^4*x+y^6,7/15*x^3-1/3*y^2*x^2-7/5*y^5*x+y^7]]
```

The first list means MList *and the second list means* SList.

Example 12
Let consider J_{10} singularity defined by $f = x^3 + y^6 + x^2y^2$. Our implementation can output a basis of H_f. The polynomial representation of a basis of H_f, is the following.

```
[469] cohomology(x^3+y^6+x^2*y^2,[x,y],1,0,0);
[[1,y,x,y^2,y*x,y^3,y^4],[-2/3*x^2+y^2*x,-3*y*x^2+9/2*y^3*x+y^5,2
*x^3-3*y^2*x^2+9/2*y^4*x+y^6]]
```

Now, we are ready to introduce applications of algebraic local cohomology (a basis of H_f). In Section 5, 6 and 7, we describe algorithms for computing standard bases, Gröbner bases and normal-form, which are applications of algebraic local cohomology.

5 Standard Bases

In this section, we propose an algorithm for computing standard bases. Let f be a polynomial in $K[x]$ such that f has an isolated singularity at the origin. Assume that $J = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$ is a zero-dimensional Jacobi ideal of f. Then, after computing a basis of H_f, we can transform the basis into a standard basis of J in $K[[x]]$. As we said, this method using algebraic local cohomology, has the following advantages.

- We do not need Mora's reduction (tangent cone algorithm) for computing standard bases.
- The algorithm ends up only with linear algebra. (We do not need difficult and heavy computational techniques.)
- The termination of the algorithm is guaranteed. This means that if we know a Jacobi ideal is a zero dimensional ideal, then the algorithm terminates and outputs a reduced standard basis.

Recall that, there is a residue pairing, denoted by $\mathrm{res}_O(\ ,\ )$, between the quotient space $K[[x]]/J$ and the vector space H_f,

$$\mathrm{res}_O(\ ,\ ) : K[[x]]/J \times H_f \longrightarrow K.$$

The pairing is nondegenerate according to the Grothendieck local duality theorem([9]). As we mentioned, the result above implies in particular that a given formal power series $p \in K[[x]]$ is in the ideal J if and only if p satisfies

$$\mathrm{res}_O(p, \psi) = 0, \ \ \forall \psi \in \mathrm{APPEND}(\mathrm{MList}, \mathrm{SList}).$$

Now, we consider the **local total degree reverse lexicographical order** on $K[[x]]$, and let $\mathrm{ht}(p)$ denote, with respect to this order, the head term of formal power series p in $K[[x]]$.

The next theorem tells us the relation between algebraic local cohomology and standard bases. By this theorem, we can construct an algorithm for computing a standard basis of J in $K[[x]]$ ([25]). In the following theorem, we regard ξ as x, and write ξ as x.

Theorem 13
In this theorem, we write $p \in \mathrm{SList}$ as $p = x^\tau + \sum_{\kappa < \tau} c_{(\tau,\kappa)} x^\kappa$. Then, the polynomial x^α for $\alpha \in \mathrm{FList}$ with $\alpha \notin \mathrm{LList}$ and $p_\alpha(x) := x^\alpha - \sum_{\kappa \in \mathrm{TList}} c_{(\kappa,\alpha)} x^\kappa$ for $\alpha \in \mathrm{set}(\mathrm{FList}) \cap \mathrm{set}(\mathrm{LList})$ give rise to the

standard basis of the ideal J with the respect to the local total degree reverse lexicographic order. Note that, if $\kappa \in$ TList, there exists q in SList which forms $q = x^\kappa + \sum_{\beta<\kappa} c_{(\kappa,\beta)} x^\beta$.

Proof It is clear from the definition that the monomial ideal in $K[[x]]$ generated by $\mathrm{ht}(f), f \in J$ coincides with that generated by x^α, $\alpha \in$ FList. That is, the list FList is the set of exponents of head terms of the standard basis. By the condition $\mathrm{res}_O(p, \phi) = 0, \forall\phi \in \mathrm{APPEND}(\mathrm{MList}, \mathrm{SList})$, it is obvious that if $\alpha \in$ FList is not in LList, then the monomial $p_\alpha(x) := x^\alpha$ itself is in the ideal J, and if $\alpha \in$ FList is in LList, then $p_\alpha(x) := x^\alpha - \sum_{\kappa\in\mathrm{TList}} c_{(\kappa,\alpha)} x^\kappa$ is also in J. ∎

(Algorithm) StandardBasis(f)

Input: f: a polynomial in $K[x]$,
Output: S: a standard basis of J in $K[[x]]$.
$S \leftarrow \{\ \}$; (MList, SList, TList, FList) $\leftarrow$ **ALC**(f)
while FList $\neq$ [ ] **do**
$\alpha \leftarrow$ CAR(FList); FList $\leftarrow$ CDR(FList)
$s \leftarrow x^\alpha - \sum_{\kappa\in\mathrm{TList}} c_{(\kappa,\alpha)} x^\kappa$ /*if $\kappa \in$ TList, $\exists q = \xi^\kappa + \sum_{\beta<\kappa} c_{(\kappa,\beta)} \xi^\beta) \in$ SList */
$S \leftarrow S \cup \{s\}$
end-while
return(S)

Example 14
Let $f = x^3y + xy^4 + x^2y^3$ be a polynomial in $K[x, y]$. In Example 8 and 10, we have already FList = $[(0,7),(1,4),(2,1),(3,0)]$, SList = $[p_1 = \eta^4 - \frac{1}{3}\xi^2\eta, p_2 = \xi\eta^3 - 4\xi^3 - \frac{2}{3}\xi^2\eta, p_3 = \eta^5 - \frac{1}{3}\xi^2\eta^2 + \xi^3, p_4 = \eta^6 - \frac{1}{3}\xi^2\eta^3 + \frac{7}{33}\xi\eta^4 + \frac{4}{3}\xi^4 + \frac{5}{33}\xi^2\eta - \frac{14}{99}\xi^2\eta^2 + \frac{14}{33}\xi^3]$ *and* TList = $[(0,4),(1,3),(0,5),(0,6)]$. *Note that, elements of corners $*$ in Figure 8-2 are in* FList, *and become head terms of a standard basis of J in $K[[x, y]]$. (We regard ξ as x, η as y.)*

(1). Take $(0,7)$ from FList. *That is, consider y^7 as the head term of a polynomial. Terms of polynomials p_1, p_2, p_3, p_4 do not have η^7. Therefore, we have y^7 as an element of a standard basis.*

(2). Take $(1,4)$ from FList. *Then, p_4 has the term xy^4 whose coefficient is $\frac{7}{33}$. As the exponent of* $\mathrm{ht}(p_4)$ *is $(0,6)$, we obtain a polynomial $xy^4 - \frac{7}{33}y^6$.*

(3). Take $(2,1)$ from FList. *Then, p_1 and p_2 have the term x^2y. The coefficient of x^2y in p_1, is $-\frac{1}{3}$ and the exponent of* $\mathrm{ht}(p_1)$ *is $(0,4)$. The coefficient of x^2y in p_2, is $-\frac{2}{3}$ and the exponent of* $\mathrm{ht}(p_2)$ *is $(1,3)$. Therefore, we obtain $x^2y + \frac{1}{3}y^4 + \frac{2}{3}xy^3$.*

(4) Take $(3,0)$ from FList. *Then, p_2, p_3 and p_4 have the term x^3. The coefficient of x^3 in p_2, is -4 and the exponent of* $\mathrm{ht}(p_1)$ *is $(1,3)$. The coefficient of x^3 in p_3, is 1 and the exponent of* $\mathrm{ht}(p_3)$ *is $(0,5)$. The coefficient of x^3 in p_4, is $\frac{14}{33}$ and the exponent of* $\mathrm{ht}(p_4)$ *is $(0,6)$. Therefore, we obtain $x^3 + 4xy^3 - y^5 - \frac{14}{33}y^6$.*

Hence, a standard basis of J is $\left\{y^7, xy^4 - \frac{7}{33}y^6, x^2y + \frac{1}{3}y^4 + \frac{2}{3}xy^3, x^3 + 4xy^3 - y^5 - \frac{14}{33}y^6\right\}$ w.r.t the local total degree reverse lexicographic term order ($1 \succ y \succ x \succ y^2 \succ xy \succ x^2 \succ y^2 \succ \cdots$).

6 Normal-form

Let f, h be polynomials and J be the Jacobi ideal of f in $K[[x]]$. In this section, we give an algorithm for computing a normal-form of h modulo an ideal generated by a standard basis of J in $K[[x]]$. **By this method, we can solve a membership problem of** J. Before describing the algorithm, we need the following two corollaries which are from Theorem 13, to construct the algorithm. Throughout this section, we fix a set $\mathrm{DL} := \mathrm{ML} \cup \mathrm{set}(\mathrm{TList}) \cup \mathrm{set}(\mathrm{LList})$, $q(x) := \sum_\lambda h_\lambda x^\lambda$ in $K[[x]]$ and J be the Jacobi ideal of a polynomial f in $K[[x]]$.

Corollary 15
Assume that $\mathrm{MList}, \mathrm{TList}, \mathrm{LList}$ *are from* $\mathbf{ALC}(f)$. *If* $\lambda \notin \mathrm{DL}$, *then,* $x^\lambda \in J$.

Corollary 16
In this corollary, we write $p \in \mathrm{SList}$ *as* $p = x^\tau + \sum_{\kappa<\tau} c_{(\tau,\kappa)} x^\kappa$. *Then, for all* $\alpha \in \mathrm{set}(\mathrm{TList}) \cup \mathrm{set}(\mathrm{LList})$, *we have the following relations* $x^\alpha \equiv \sum_{\kappa\in\mathrm{TList}} c_{(\kappa,\alpha)} x^\kappa \bmod J$. *Note that, if* $\kappa \in \mathrm{TList}$, *there exists* q *in* SList *which forms* $q = x^\kappa + \sum_{\beta<\kappa} c_{(\kappa,\beta)} x^\beta$.

By these two corollaries and Theorem 13, we can construct an algorithm for computing normal-form by using algebraic local cohomology.

(Algorithm) Normal-form(f, h)

Input: f, q: polynomials in $K[x]$, (J is the Jacobi ideal of f.)
Output: g: a normal-form of q modulo J in $K[[x]]$ (i.e., $q \equiv g \bmod J$).

(0) $(\mathrm{MList}, \mathrm{SList}, \mathrm{TList}, \mathrm{FList}, \mathrm{LList}) \leftarrow \mathbf{ALC}(f)$
$\mathrm{ML} \leftarrow \{\deg(t) | t \in \mathrm{MList}\}$; $\mathrm{DL} \leftarrow \mathrm{ML} \cup \mathrm{set}(\mathrm{TList}) \cup \mathrm{set}(\mathrm{LList})$

(1) Decompose $q(x)$ as $q(x) = \sum_{\lambda\in\mathrm{DL}} q_\lambda x^\lambda + \sum_{\lambda'\notin\mathrm{DL}} q_{\lambda'} x^{\lambda'}$.
By Corollary 15, $q(x) \equiv \sum_{\lambda\in\mathrm{DL}} q_\lambda x^\lambda \bmod J$.

(2) Decompose $\sum_{\lambda\in\mathrm{DL}} q_\lambda x^\lambda = \sum_{\tau\in\mathrm{ML}} q_\tau x^\tau + \sum_{\gamma\in\mathrm{DL}\setminus\mathrm{ML}} q_\gamma x^\gamma$.

(3) By Corollary 16, $\forall\alpha \in \mathrm{set}(\mathrm{TList}) \cup \mathrm{set}(\mathrm{LList})$, $x^\alpha \equiv \sum_{\kappa\in\mathrm{TList}} c_{(\kappa,\alpha)} x^\kappa \bmod J$.
Transform the second part $\sum_{\gamma\in\mathrm{DL}\setminus\mathrm{ML}} q_\gamma x^\gamma$ into $\sum_{\tau\in\mathrm{TList},\ \gamma\in\mathrm{DL}\setminus\mathrm{ML}} c_{(\gamma,\tau)} x^\tau$ by the relations.

(4) We obtain $g = \sum_{\tau\in\mathrm{ML}} q_\tau x^\tau + \sum_{\tau\in\mathrm{TList},\ \gamma\in\mathrm{DL}\setminus\mathrm{ML}} c_{(\gamma,\tau)} x^\tau$ such that $q(x) \equiv g \bmod J$.
return(g)

Example 17
Let $f = x^3y + xy^4 + x^2y^3$ *be a polynomial in* $K[x, y]$. *In Example 8 and 10, we already have* $\mathrm{MList} = [1, \xi, \xi^2, \eta, \xi\eta, \eta^2, \xi\eta^2, \eta^3]$, $\mathrm{SList} = [p_1 = \eta^4 - \frac{1}{3}\xi^2\eta, p_2 = \xi\eta^3 - 4\xi^3 - \frac{2}{3}\xi^2\eta, p_3 = \eta^5 - \frac{1}{3}\xi^2\eta^2 + \xi^3, p_4 = \eta^6 - \frac{1}{3}\xi^2\eta^3 + \frac{7}{33}\xi\eta^4 + \frac{4}{3}\xi^4 + \frac{5}{33}\xi^2\eta - \frac{14}{99}\xi^2\eta^2 + \frac{14}{33}\xi^3]$, $\mathrm{TList} = [(0, 4), (1, 3), (0, 5), (0, 6)]$ *and* $\mathrm{LList} = [(2, 3), (1, 4), (4, 0), (3, 1), (2, 2), (3, 0), (2, 1)]$. *Hence,* $\mathrm{DL} = [(6, 0), (2, 3), (1, 4), (0, 5), (4, 0), (3, 1), (2, 2), (1, 3), (0, 4), (3, 0), (2, 1), (0, 3), (1, 2), (0, 2), (1, 1), (0, 1), (2, 0), (1, 0), (0, 0)]$. *And, by Corollary 16, there exist relations* $x^2y^3 \equiv -\frac{1}{3}y^6 \bmod J$, $xy^4 \equiv \frac{7}{33}y^6 \bmod J$, $x^4 \equiv \frac{4}{3}y^6 \bmod J$,

$x^3y \equiv \frac{5}{33}y^6 \bmod J$, $x^2y^2 \equiv -\frac{1}{3}y^5 - \frac{14}{99}y^6 \bmod J$, $x^3 \equiv -4xy^3 + y^5 + \frac{14}{33}y^6 \bmod J$ *and* $x^2y \equiv -\frac{1}{3}y^4 - \frac{2}{3}xy^3 \bmod J$. *Consider two polynomials* $h_1 = 4x^3y^2 + 5x^2y + 2x + y$ *and* $h_2 = x^4 - 6x^3y - 2xy^4$. *First, we consider a normal-form of* h_1 *by a standard basis of* J. *Since* $(3,2) \notin \mathrm{DL}$, $(0,1),(1,0) \in \mathrm{ML}$ *and* $(2,1) \in \mathrm{LList}$, *we obtain* $h_1 \equiv 5x^2y + 2x + y \bmod J$. *Furthermore, as* $x^2y \equiv -\frac{1}{3}y^4 - \frac{2}{3}xy^3 \bmod J$, *we obtain* $h_1 \equiv 5(-\frac{1}{3}y^4 - \frac{2}{3}xy^3) + 2x + y \bmod J$. *Therefore,* h_1 *can be reduced to* $y + 2x - \frac{5}{3}y^4 - \frac{10}{3}xy^3$ *by a standard basis of* J.

Next, we consider a normal-form of h_2 *by the standard basis of* J. *Since* $(4,0),(3,1),(1,4) \in \mathrm{LList}$ *and* $x^4 \equiv \frac{4}{3}y^6, x^3y \equiv \frac{5}{33}y^6, xy^4 \equiv \frac{7}{33}y^6$, *we obtain* $h_2 \equiv \frac{4}{3}y^6 - 6(\frac{5}{33}y^6) - 2(\frac{7}{33}y^6) = 0$. *Therefore, this means that* h_2 *is a member of* J *in* $K[[x]]$ *(i.e.,* $h_2 \in J \subset K[[x]]$*).*

7 Gröbner Bases

In this section, we describe an algorithm for computing Gröbner bases. Theorem 13 says that once one has a basis of H_f (algebraic local cohomology classes), one can directly compute a standard basis of a Jacobi ideal from the basis. One can also derive an algorithm the primary component supported at the origin of the zero-dimensional ideal. Let I be a zero-dimensional ideal with an affine variety $\mathbb{V}(I) = \{p_1, \ldots, p_m\}$. Assume that $Q_1 \cap \cdots \cap Q_m$ is the primary decomposition of I in $K[x]$ where Q_i is a primary ideal supported at the p_i. Then, it is well-known that $K[x]/I \cong K[x]/Q_1 \times \cdots \times K[x]/Q_m$ (see [5]). By this fact and the definition of H_f, one can easily improve the algorithm StandardBasis to compute a Gröbner basis of Q_i where Q_i is the primary component supported at the origin. **In this section, we write this Q_i as J_O. Note that our algorithm is free from primary decomposition algorithm.** The algorithm for computing Gröbner bases using algebraic local cohomology, is the following.

(Algorithm) GröbnerBasis(f)

Input: f: a polynomial in $K[x]$, $\prec$: a global term order,
Output: S: a Gröbner basis of J_O w.r.t. $\prec$.
(0). $S \leftarrow \{\ \}$; (MList, SList, TList, LList, FList) $\leftarrow$ **ALC**(f); $[\alpha_1, \ldots, \alpha_t] \leftarrow$ Line up all elements of APPEND(TList, LList) in order of $\prec$, where $\xi^{\alpha_1} \prec \cdots \prec \xi^{\alpha_t}$.
(1). Let $[p_1, \ldots, p_m]$ = SList. Make the coefficient matrix Φ of SList.
I.e., $\begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} = \Phi \begin{pmatrix} \xi^{\alpha_1} \\ \vdots \\ \xi^{\alpha_t} \end{pmatrix}$. $\Phi' \leftarrow$ Compute the row reduced echelon matrix of Φ.
(2). $\begin{pmatrix} q_1 \\ \vdots \\ q_m \end{pmatrix} \leftarrow \Phi' \begin{pmatrix} \xi^{\alpha_1} \\ \vdots \\ \xi^{\alpha_t} \end{pmatrix}$ TL $\leftarrow \{\deg(q_1), \ldots, \deg(q_m)\}$
    ML $\leftarrow \{\deg(g) | g \in \text{MList}\}$
(3). $G \leftarrow$ ReducedGröbnerBasis($\langle \xi^\gamma | \forall \gamma \in \mathrm{nb}(\mathrm{TL} \cup \mathrm{ML}) \setminus (\mathrm{TL} \cup \mathrm{ML}) \rangle$).
    KList $\leftarrow$ list($\{\deg(\tau) \in \mathbb{N}^n | \ \forall \tau \in G\}$) /* In Figure 18, "$*$". */
(4). while KList $\neq$ [ ] **do**
  $\sigma \leftarrow$ CAR(KList); KList $\leftarrow$ CDR(KList)
  $s \leftarrow x^\sigma - \sum_{\kappa \in \mathrm{TL}} c_{(\kappa,\gamma)} x^\kappa$ /*if $\kappa \in$ TL, then $\exists q = x^\kappa + \sum_{\beta \prec \kappa} c_{(\kappa,\beta)} x^\beta \in \{q_1, \ldots, q_m\}$ */
  $S \leftarrow S \cup \{s\}$
end-while
return(S)

Example 18
Let $f = x^3y + xy^4 + x^2y^3$ be a polynomial in $K[x, y]$. Consider the global total degree lexicographic order $\prec$ such that $y \prec x$. Let G_O be a Gröbner basis of J_O where J_O is the primary component of the Jacobi ideal of f, supported at the origin. We consider to obtain G_O. In Example 8 and 10, we have already SList $= [p_1 = \eta^4 - \frac{1}{3}\xi^2\eta, p_2 = \xi\eta^3 - 4\xi^3 - \frac{2}{3}\xi^2\eta, p_3 = \eta^5 - \frac{1}{3}\xi^2\eta^2 + \xi^3, p_4 = \eta^6 - \frac{1}{3}\xi^2\eta^3 + \frac{7}{33}\xi\eta^4 + \frac{4}{3}\xi^4 + \frac{5}{33}\xi^2\eta - \frac{14}{99}\xi^2\eta^2 + \frac{14}{33}\xi^3]$, TList $= [(0, 4), (1, 3), (0, 5), (0, 6)]$ *and* LList $= [(2, 3), (1, 4), (4, 0), (3, 1), (2, 2), (3, 0), (2, 1)]$.

We line up all elements of APPEND(TList, LList) *in order of $\prec$ as follows:*

$$(2, 1), (3, 0), (0, 4), (1, 3), (2, 2), (3, 1), (4, 0), (0, 5), (1, 4), (2, 3), (0, 6).$$

Next, we make the coefficient matrix Φ of SList.

$$\Phi = \begin{array}{c} \begin{matrix} \xi^2\eta & \xi^3 & \eta^4 & \xi\eta^3 & \xi^2\eta^2 & \xi^3\eta & \xi^4 & \eta^5 & \xi\eta^4 & \xi^2\eta^3 & \eta^6 \end{matrix} \\ \begin{pmatrix} -\frac{1}{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{3} & -4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{14}{33} & 0 & 0 & -\frac{14}{99} & \frac{5}{33} & \frac{4}{3} & 0 & \frac{7}{33} & -\frac{1}{3} & 1 \end{pmatrix} \end{array}$$

The next matrix Φ' is in reduced row echelon form of Φ.

$$\Phi' = \begin{array}{c} \begin{matrix} \xi^2\eta & \xi^3 & \eta^4 & \xi\eta^3 & \xi^2\eta^2 & \xi^3\eta & \xi^4 & \eta^5 & \xi\eta^4 & \xi^2\eta^3 & \eta^6 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & -\frac{3}{2} & 2 & 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{2}{3} & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{44}{5} & -\frac{14}{5} & \frac{7}{5} & -\frac{11}{5} & \frac{33}{5} \end{pmatrix} \end{array}$$

We have ML $= \{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 0), (2, 1), (3, 0)\}$,TL $= \{(3, 0), (2, 1), (3, 1), (4, 1)\}$. *In Figure 18, $\bullet$ means an element of* ML *and $\triangle$ means an element of* TL. *Now, we can compute the reduced Gröbner basis G of $\langle \xi^\gamma | \gamma \in \mathrm{nb}(\mathrm{TL} \cup \mathrm{ML}) \backslash (\mathrm{TL} \cup \mathrm{ML}) \rangle$. Then, $G = \{\xi^5, \xi\eta^3, \xi^2\eta^2, \xi^4\}$ and* KList $= \{(0, 5), (1, 3), (2, 2), (4, 0)\}$. *In Figure 18, $*$ means an element of* KList. *By Corollary 16, we have the following relations*

$$\begin{cases} xy^3 \equiv -\dfrac{3}{2}x^2y - \dfrac{1}{2}, \\ x^2y^2 \equiv 2x^2y - \dfrac{1}{3}x^3 + \dfrac{2}{3}y^4, \\ x^4 \equiv \dfrac{44}{5}x^3y, \\ y^5 \equiv -6x^2y + x^3 - 2y^4 - \dfrac{12}{5}x^3y, \quad \bmod J_O \\ xy^4 \equiv \dfrac{7}{5}x^3y, \\ x^2y^3 \equiv -\dfrac{11}{5}x^3y, \\ y^6 \equiv \dfrac{33}{5}x^3y. \end{cases}$$

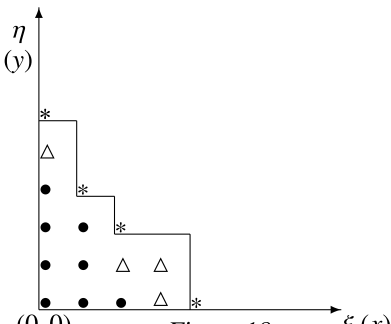

Figure 18

Now, we can follow the same procedure of the algorithm StandardBasis *to compute a Gröbner basis G_O of J_O w.r.t. $\prec$. Then, we obtain the following set G_O of polynomials as the Gröbner basis;* $G_O = \left\{xy^3 + \frac{3}{2} + \frac{1}{2}y^4, x^2y^2 - 2x^2y + \frac{1}{3} - \frac{2}{3}y^4, x^4 - \frac{44}{5}x^3y, y^5 \ +6x^2y - x^3 + 2y^4 + \frac{12}{5}x^3y\right\}$.

Example 19
Let consider W_{25} singularity defined by $f = x^4 + xy^7 + x^2y^5$. Our implementation outputs a Gröbner basis of J_O w.r.t. the global total degree lexicographic order s.t. $x > y$, which is the following.

```
[x^6,y*x^5,5/7*x^5+y^2*x^4,-196/45*y*x^4+y^4*x^3,
28/9*x^4+y^5*x^2,5/7*y^4*x^2+y^6*x,4*x^3+2*y^5*x+y^7]
```

8 Comparison

All algorithms in this paper have been implemented by the author in the computer algebra system Risa/Asir. In fact, the author's program (NEW) of standard bases and normal-form, is unique one. Abe has not implemented them. In this final section, we compare Abe's program ([1]) to our program. Both programs work on the computer algebra system Risa/Asir. We measure the computational time of both programs. In this comparison, both programs execute for obtaining a basis of H_f (algebraic local cohomology). We use the following computation environment [CPU: Pentium M 1.73 GHZ, OS: Windows XP] and Risa/Asir version 20080904 (Kobe Distribution). In the table, x, y, z are variables.

f		Abe	NEW
$x^3y + xy^4 + x^2y^3$	(Milnor no. 12)	0.11	0.015
$(x^2 + y^3)^2 + (2 + 3y)x^2y^3$	(Milnor no. 15)	0.141	0.015
$(x^2 + y^3)^2 + (2 + 3y)x^2y^5$	(Milnor no. 19)	0.25	0.063
$x^4 + xy^7 + x^2y^5$	(Milnor no. 25)	0.282	0.031
$(x^2 + y^3)^2 + (2 + 3y)x^2y^{10}$	(Milnor no. 29)	2.235	1.531
$x^4 + x^3y^3 + x^2y^5 + y^{12}$	(Milnor no. 29)	0.375	0.062
$x^2y + z^3y + 3y^2z^2 + y^4 + z^{14}$	(Milnor no. 35)	6.594	3.969
$(x^2 + y^3)^2 + (2 + 3y)x^2y^{20}$	(Milnor no. 49)	84.99	64.19
$x^2z + yz^2 + y^22 + 3xy^{17} + 8zy^{12}$	(Milnor no. 65)	85.98	10.05
$(x^2 + y^3)^2 + (2 + 3y)x^2y^{30}$	(Milnor no. 69)	1285	683.3
$x^4 + x^3y^9 + x^2y^{17} + y^{44}$	(Milnor no. 109)	31.97	4.015
$x^2z + yz^2 + y^{42} + 3xy^{32} + 8zy^{22}$	(Milnor no. 125)	> 2 hours	113.3

(CPU sec.)

As solving a system of linear equations is costly, we need a small number of candidates of lower terms, to compute a basis of H_f, efficiently. Since in this point, the author's implementation has a big advantage, the implementation is more efficient than Abe's one.

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